

QUANTUM FIELD THEORY, EFFECTIVE POTENTIALS AND DETERMINANTS OF ELLIPTIC OPERATORS

A Thesis Submitted to the
College of Graduate Studies and Research
in Partial Fulfillment of the Requirements
for the degree of Master of Science
in the Department of Physics & Engineering Physics
University of Saskatchewan
Saskatoon

By
Percy L. Paul

©Percy L. Paul, April 2010. All rights reserved.

PERMISSION TO USE

In presenting this thesis in partial fulfilment of the requirements for a Postgraduate degree from the University of Saskatchewan, I agree that the Libraries of this University may make it freely available for inspection. I further agree that permission for copying of this thesis in any manner, in whole or in part, for scholarly purposes may be granted by the professor or professors who supervised my thesis work or, in their absence, by the Head of the Department or the Dean of the College in which my thesis work was done. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to the University of Saskatchewan in any scholarly use which may be made of any material in my thesis.

Requests for permission to copy or to make other use of material in this thesis in whole or part should be addressed to:

Head of the Department of Physics & Engineering Physics
115 Science Place
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 5E2, Canada

ABSTRACT

The effective potential augments the classical potential with the quantum effects of virtual particles, and permits the study of spontaneous symmetry breaking. In contrast to the standard approach where the classical potential already leads to electroweak symmetry-breaking, the Coleman-Weinberg mechanism explores quantum corrections as the source of symmetry-breaking. This thesis explores extensions of the Coleman -Weinberg mechanism to the situations with more than one Higgs doublet. These multi-Higgs models have a long history [61], and occur most naturally in the Minimal Supersymmetric model. Mathematical foundations of the zeta function method will be developed and then applied to regularise the one-loop computation of the effective potentials in a model with two scalar fields.

ACKNOWLEDGEMENTS

I would like to acknowledge first Tom for greatest support throughout this project. He has helped a person of my temperament in ways that are unimaginable. His students also helped in ways away from the office. Thanks Robin.

Special Thanks to J. and M. Mitchell for years of awesome support without whom I would be elsewhere and not in math and physics. You guys are the best.

Thanks to the following in no order: Jessica, Mary Jane, Mother Mary Jane, Beskkaai, Chevez, Caitlin, La Raine, Michele, Melissa, Lynne, Don, Veronica, Rod, Chuck, Chantal, Curtis, Jane, members of the English River Dene Nation.

I am grateful to the committee for comments and questions: Dr. C. Xiao , Dr. Jacek Szmigielski, Dr. R. Dick, Dr. K. Tanaka, Dr. R. Pywell.

“...there may be some basic flaw in our whole approach which we have been too stupid to see.” Ref. [27]

To the Memory of Ida L. Paul and Ryan Apeis

CONTENTS

Permission to Use	i
Abstract	ii
Acknowledgements	iii
Contents	v
List of Tables	vii
List of Figures	viii
1 Introduction	1
2 Quantum Field Theory	7
2.1 Functional Integral and Quantisation	9
2.1.1 Connected Green Functions	10
2.1.2 Generating Functional for One-Particle Irreducible Green Function	11
2.2 Effective Potentials for Massless Scalar Fields	13
2.2.1 Background Field Method in Quantum Field Theory	14
3 Determinants, Differential Operators and Vector Bundles	17
3.1 The Geometry of Gauge Field Theories	17
3.1.1 Vector Bundles, Sections, and Connections	17
3.1.2 Lie Groups, Lie Algebras and Symmetry	18
3.1.3 Principal Bundles	18
3.2 Differential Operators	21
3.3 Heat Kernel Constructions	23
3.4 Riemann zeta Function	23
3.5 Functional Determinants	24
3.5.1 Determinants	24
3.5.2 Functional Determinants on Differential operators	25
4 The Standard Model of Particles	26
4.1 Quantising the Gauge Theories	28
4.2 Electroweak Theory or $SU(2)_L \times U(1)_Y$ - Bundle	29
4.3 Higgs Mechanism and Spontaneous Symmetry Breaking	29
4.3.1 Massive Gauge Potential	31
4.4 Electroweak Symmetry Breaking	34
4.4.1 Spontaneous Symmetry Breaking	34
4.4.2 Chiral Fermions, Yukawa Interactions and Mass Generation . . .	40

4.4.3	Two Higgs Doublet Models	42
5	Radiative Electroweak Symmetry Breaking	44
5.1	Coleman-Weinberg via zeta Function Regularisation	44
5.1.1	Loop expansion of the Effective Potential	45
5.2	Two Scalar Fields with the Identical Method	52
5.3	A Simple not-so-Simple Model	53
6	Applications of zeta Function Method	59
6.1	Coleman-Weinberg in Type 0B String theory	59
6.2	Gross-Neveu Model	60
6.3	Standard Model Effective Potential	62
6.4	Super-Feynman Rules	63
6.5	Supersymmetry	64
6.5.1	Superspace and Superfields	64
6.6	Chiral and Vector Multiplets and Superfields	67
6.6.1	Chiral Superfields	67
6.7	Minimal Supersymmetric Standard Model	68
7	Conclusion	69
	References	71

LIST OF TABLES

1.1	The particle content in the $SU(3)_c \times SU(2)_L \times U(1)_Y$ theory	4
-----	---	---

LIST OF FIGURES

- 2.1 The one-loop contributions to the effective potential. The dots indicate the insertion of zero-momentum external fields appropriate for the model. . . . 14

CHAPTER 1

INTRODUCTION

Nuclear β -decay was the first observed weak interaction process. Pauli postulated the existence of a neutrino to conserve energy in β decays such as $n \rightarrow p + e + \bar{\nu}$. These weak interactions were very short range, and corresponded to comparatively long lifetimes compared to strong nuclear decays. Because β decays involve four spin-1/2 particles, the “four-Fermi” theory of β decay was developed by Fermi. From a modern perspective, the four-Fermi theory must include all the known experimental features of the weak interactions: parity violation, neutral and charged-current interactions, and universality of interaction strength for quarks and leptons.

The first observation of parity violation was in the K^+ decays $K^+ \rightarrow \pi^+\pi^+\pi^0$ and $K^+ \rightarrow \pi^+\pi^0$. Since the final states have different parity, it means that the interaction responsible for these decays must violate parity. Later it was found that weak interactions violate parity in the maximum possible way, producing only left-handed particle states. The left-handed nature of the weak interactions means that the four-Fermi theory must have a “V-A” (vector minus axial vector) Lorentz structure. Neutral current interactions were first discovered in a purely leptonic flavour-conserving elastic process: $\bar{\nu}_\mu e \rightarrow \bar{\nu}_\mu e$. Similar reactions can occur for quarks. Thus the four-Fermi theory must include the possibility of products of neutral currents as well as the charged currents occurring in β decay.

Universality can be seen in the two reactions: muon decay $\mu \rightarrow e\bar{\nu}_e\nu_\mu$ and $d \rightarrow ue\bar{\nu}_e$ (*i.e.*, the quark process underlying β decay). The coupling constants for both of these

processes are approximately equal. This gives credence to the universality of weak interactions. Any differences are the result of flavour rotations (*e.g.*, Cabbibo angle) amongst the quark flavours.

The four-Fermi theory is not a renormalisable theory because it contains an expansion constant with dimension inverse mass squared. Physically, renormalisation is a strict property for a theory to have predictive ability (such as quantum electrodynamics). This means that another theory of weak interactions is needed.

The combination of parity violation and universality suggests the existence of a gauge theory with a left-handed symmetry group along with massive intermediate vector bosons corresponding to the short-range nature of the weak interactions. Unfortunately, the massive vector bosons also lead to a non-renormalisable theory.

The best candidate for introducing massive vector bosons is the Higgs mechanism. This introduces a scalar particle into the theory that has a non-zero vacuum expectation value which is a source of spontaneous symmetry breaking. This gives mass to the vector bosons and other particles in the theory while maintaining renormalisability. This results in three weak vector bosons that mediate the weak interaction. They are W^\pm which are charged and the Z^0 which is neutral under electromagnetism. Charged current weak interactions involve the exchange of a virtual W^\pm while the neutral current interactions involve the exchange of a virtual Z^0 .

“From a certain distance there is less cause for astonishment; the concepts of space and symmetry are so fundamental that they are necessarily central to any serious scientific reflection. Mathematicians as influential as Bernhard Riemann or Hermann Weyl, to name only a few, have undertaken to analyze these concepts on the dual levels of mathematics and physics.” [1]

The underlying symmetry of the spontaneously-broken theory is necessary for renor-

malisability. This results in a unified theory of weak and electromagnetic interactions that details the structure of most known particles to date and as such represents the most successful theory of fundamental interactions. Its simple structure makes it even more attractive. However, as with most theories it is fraught with problems; the hierarchy problem (*e.g.*, the large discrepancy between the weak scale and unification scale) is one of them. Another problem is the large number of parameters which are seemingly arbitrary (such as the wide range of lepton masses) needed to specify the theory.

All of non-gravitational interactions of the particles we have seen so far can be explained by a quantum gauge theory with the symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y$ (c is colour, L is left handed and Y is hypercharge) which is broken spontaneously to $SU(3)_c \times U(1)_{em}$. When this symmetry is gauged, we end up with eight gauge bosons of strong interaction (gluons, G^a , $a = 1 \dots 8$), three intermediate weak bosons (W^a , $a = 1, 2, 3$), and an abelian boson B which is a linear combination of physical photon and neutral weak boson Z^0 . The action of the gauge group $SU_L(2)$ is on left-handed spinors corresponding to maximal parity violation observed in weak interactions. Some of these gauge bosons acquire a mass to give the short range of the weak force. The Higgs boson Φ is introduced to generate masses in the theory via a symmetry breaking mechanism.

In classical physics, particles are thought of as one-dimensional submanifolds of a four dimensional manifold which have timelike properties. Mass has various definitions in general relativity such as the ADM (Arnowitz, Deser and Misner) approach [41]. The test particle is put at asymptotes to see how it behaves under the gravitational field and compared with Newtonian potential. The ADM [41] mass is like the charge and an integration over spacetime of a quantity all the while assuming that it remains positive. The way a mass is defined in quantum field theory is as the coefficient of quadratic terms in the Lagrangian and is a pole of the two point function. Electroweak theory is a renormalisable

Fermionic Fields	Y- Hypercharge	$SU(2)_L$ Representations	$SU_c(3)$
$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{3}$	2	3
u_R	$\frac{4}{3}$	1	3
d_R	$-\frac{2}{3}$	1	3
$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	-1	2	1
e_R	-2	1	1
Bosonic Fields	Y- Hypercharge	$SU(2)_L$ Representations	$SU_c(3)$
$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	1	2	1
$T^a W_\mu^a$	0	3	1
B_μ	0	1	1
G_μ^a	0	1	8

Table 1.1: The particle content in the $SU(3)_c \times SU(2)_L \times U(1)_Y$ theory

quantum gauge theory which is based on a principal fibre bundle¹ with structure group $SU(2) \times U(1)$ and left-right asymmetric (parity violation) so the $SU(2)$ is coupled only to the left-handed projections and denoted $SU(2)_L$.

Left-handed fermions are doublets in $SU(2)$ where as the right-handed fermions are singlets. The electric charge is defined as the operator $T^3 + Y/2$, where the $1/2$ is that of the $g'/2$ coupling to the left handed particle content. The other way to describe this theory is at the tangent of the Lie groups, the Lie algebra representation is a vector space making it into a vector bundle since the space is a linear vector space of complex fields. The Lagrangian density for the strong and electroweak interactions is

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs}, \quad (1.1)$$

¹The most familiar example of a fibre bundle is the (vector and scalar) potentials in electromagnetic theory.

where²

$$\begin{aligned}
\mathcal{L}_{gauge} = & -\frac{1}{2}\text{Tr } G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\text{Tr } W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\
& + i\bar{L}_\alpha\gamma^\mu D_\mu L_\alpha + i\bar{Q}_\alpha\gamma^\mu D_\mu Q_\alpha + i\bar{e}_\alpha\gamma^\mu D_\mu e_\alpha \\
& + i\bar{u}_\alpha\gamma^\mu D_\mu u_\alpha + i\bar{d}_\alpha\gamma^\mu D_\mu d_\alpha + (D_\mu\Phi)^\dagger(D_\mu\Phi),
\end{aligned} \tag{1.2}$$

and all field multiplets are defined in Table 1.1. The mass terms, which would require a product of left- and right-handed components, cannot appear as they would violate gauge invariance.

The gauge covariant derivatives, D_μ , act on the field very differently depending on the “charge”. By charge I will mean, for example, the parameter appearing in the minimal coupling caused by the gauge symmetry such as $U(1)$ in the case of electric charge. The object $G_{\mu\nu}$ is called the *curvature* of the connection G_μ^a and similarly with other fields. Quark doublets are represented by Q (See Table 1.1), and left-handed leptons by L and the lower case e , u and d are right-handed singlets under $SU(2)$. The quark field being a doublet in weak symmetry and charged with colour means that the covariant derivative is on all connections. Singlets are immune to the actions of the Lie groups.

The complex doublet Φ has a gauge invariant Lagrangian³

$$\mathcal{L}_{Higgs} = -V = \mu^2\Phi^\dagger\Phi - \frac{\lambda}{2}(\Phi^\dagger\Phi)^2. \tag{1.3}$$

The gauge invariant Yukawa interaction is, using $\Phi_c \equiv -i\sigma_2\Phi^*$,

$$\mathcal{L}_{Yukawa} = y_{\alpha\beta}^d\bar{Q}_\alpha d_\beta\Phi + y_{\alpha\beta}^L\bar{L}_\alpha e_\beta\Phi + y_{\alpha\beta}^u\bar{Q}_\alpha u_\beta\Phi_c + \text{h.c.}, \tag{1.4}$$

²The Einstein summation convention for repeated indices is used in this thesis unless indicated otherwise.

³Note that we do not repeat the Higgs kinetic term $(D_\mu\Phi)^2$ because it would lead to a double counting in (1.1).

and this is a singlet in $SU(2)$ and Φ_c is needed for this choice.

The above Lagrangian will have the following 17 free parameters:

- The coupling constants: g_s for colour, g for $SU_L(2)$ weak and g' for the $U_Y(1)$ Abelian coupling.
- Higgs coupling λ .
- Higgs mass parameter μ^2 .
- Yukawa matrices y^d , y^L and y^u replicated for three generations.

With these parameters one can specify how a scalar Higgs gives rise to experimentally consistent quantities. One way of reducing the number of free parameters is to embed the Standard Model within a unified theory with a larger-rank symmetry group with additional fields. In the case of interest in many areas of theoretical physics is the inclusion of more than one Higgs fields, known as extended Higgs sectors such as the minimal supersymmetric Standard Model (MSSM). The Standard Model can be thought of as a limit to such theories with limits obtained by supersymmetry breaking of which there are many ways, and even more with introduction of D-branes in string theory. In this thesis only the Higgs sector will be analyzed in detail, although the method is ubiquitous in most one loop computations.

This thesis will conclude that symmetry breaking can occur radiatively. This can be shown by computing the effective potential that results in a vacuum expectation value of scalar fields. This is a quantum effect because it requires loop-level corrections. The method used is the zeta function regularisation and we get a result in agreement with Feynman diagrammatic method which verifies the zeta function techniques used.

CHAPTER 2

QUANTUM FIELD THEORY

Quantum field theory (see [11, 66, 63] for reviews) allows computation of physical quantities (such as scattering amplitudes) in high-energy regimes. It is an intersection of ideas of special relativity and quantum mechanics. It works for high speeds close to the speed of light and at the very short distances of atoms and shorter. There are different methods of how to obtain a quantum theory or to quantise a classical theory. One is the canonical quantisation where one employs the Heisenberg equations for the classical canonical field variables. In order to canonically quantise a field theory, one uses equal-time commutation relations and particles are then defined as states resulting from operators acting on a vacuum state.

Another method to quantise a theory is the path integral method which I shall employ. This method realizes symmetries of the Lagrangian explicitly in the notation. Therefore path integrals are the best way to study the quantum effects when symmetries are important.

Fields in most quantum field theories are sections of bundles. Since most describe particle physics we will need spin bundles on spin manifolds. However, I will only describe the vector bundles associated with Higgs particle.

As discussed in more detail below, there are three basic Green functions used in quantum field theory

- Full n -point Green functions $G^{(n)}(x_1, \dots, x_n)$ obtained from the generating func-

tional $Z[J]$

- Connected Green functions $G_{conn}^{(n)}(x_1, \dots, x_n)$ obtained from quantum action $W[J]$
- Proper Green functions $\Gamma^{(n)}(x_1, \dots, x_n)$ obtained from $\Gamma[\phi]$

These sets of Green functions could be illustrated as: $\Gamma \subset W \subset Z$.

Later (see Eq. (5.11)) I will obtain the following formula for the effective action

$$S[\phi_c, J] = \int [d\phi] e^{S[\phi, J]} = -\frac{1}{2} \ln \det[A], \quad (2.1)$$

where A is related to the quadratic part of the action; I will discuss some of its consequences.

There are a few problems with this formal expression and I shall clarify the results. The actions are the effective actions obtained by integrating out the large modes, that is by integrating over quantum fluctuations about a classical background. This approach is known as the background field method and the origins of radiative symmetry has this to credit as viability.

Here I shall only be concerned with functional integration or path integral methods in quantum field theory [17, 66, 12, 63]. These are used to obtain Green Functions: *e.g.*, for the scalar field ϕ ,

$$G^{(n)}(x_1 \dots x_n) = \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle \quad (2.2)$$

which can be given in terms of a partition function $Z[J]$ of a weighted integral called the functional integral or path integral.

I shall however bypass the usual starting point on most discussions on functional integral which is through quantum mechanics and sum over histories so the formalism is quickly obtained. I go directly to a field theory version.

2.1 Functional Integral and Quantisation

The action is the usual starting point for many field theories applied to particle physics. The action can have both global and local symmetries, which leave the action of the theory invariant under some Lie group action. In four dimensions, the action is represented as an integral of a 4-form. Classical fields are replaced by field operators. The typical illustrative example used in quantum field theory literature is the ϕ^4 theory whose Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4, \quad (2.3)$$

has a discrete symmetry $\phi \rightarrow -\phi$. This action has as its classical equation of motion

$$(\partial^\mu \partial_\mu + m^2) \phi = -\frac{\lambda}{6} \phi^3. \quad (2.4)$$

The vacuum-to-vacuum transition amplitude in the presence of a source $J(x)$ is

$$\begin{aligned} Z[J] &= \int [d\phi] \exp i \int d^4x \left[\mathcal{L}(\phi, \partial\phi) + J\phi + \frac{1}{2} i \varepsilon \phi^2 \right] \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G^{(n)}(x_1 \dots x_n) J(x_1) \dots J(x_n) \end{aligned} \quad (2.5)$$

The term containing ε ensures convergence of the path integral and the $\varepsilon \rightarrow 0^+$ limit is implicit; alternatively, this term can be omitted and the theory can be considered in a Euclidean space. This quantity is an integral over fields ϕ and it has been difficult if not impossible to define mathematically. So I simply assume that the path integral exists and is useful. We will mostly work on a compact Euclidean space, but for this section I'll stay

with Minkowski spacetime. The n -point Green functions of the theory can be written as

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{1}{i^n} \frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)} \Big|_{J=0}. \quad (2.6)$$

For free-fields it is possible for us to put the Lagrangian into a quadratic form using

$$\int \partial_\mu \phi \partial^\mu \phi d^4x = - \int \phi \partial_\mu \partial^\mu \phi d^4x = - \int \phi \partial^2 \phi d^4x,$$

where $\partial^2 = \partial_\mu \partial^\mu$. Then

$$Z_0[J] = \int [d\phi] \exp \left(-i \int d^4x \left[\frac{1}{2} \phi (\partial^2 + m^2 - i\varepsilon) \phi - J\phi \right] \right), \quad (2.7)$$

which is the free field functional integral and gives the Green Function for ϕ .

2.1.1 Connected Green Functions

The Feynman diagrams for connected Green Functions are topologically connected graphs.

This means one can make full Green functions out of them. Define $G_c^{(n)}$ to be the connected part of $G^{(n)}$

$$W[J] = \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int d^4x_1 \dots d^4x_n G_c^{(n)}(x_1 \dots x_n) J(x_1) \dots J(x_n) \quad (2.8)$$

They are related by the formal expression which is the definition of the quantum action

$W[J]$,

$$Z[J] = \exp [iW[J]]. \quad (2.9)$$

2.1.2 Generating Functional for One-Particle Irreducible Green Function

The books [70, 63] are excellent sources for these topics and I shall follow their discussion closely. A one-particle irreducible (1PI) Green function comes from computing a one particle irreducible Feynman diagram. As discussed below, these 1PI processes give us the quantum corrections to the classical Lagrangian. Masses in general will be defined as isolated poles of a two-point Green function [65, 63]

Consider the generating functional for the scalar field ϕ , with a source $J(x)$ added at will to the Lagrangian, and the vacuum to vacuum amplitude

$$\langle 0, \infty | 0, -\infty \rangle^J = Z[J] = \int \mathcal{D}\phi \exp \left(i \int d^4x [\mathcal{L}(\phi) + J(x)\phi(x)] \right), \quad (2.10)$$

which is the vacuum-to-vacuum amplitude at the asymptotic regions of spacetime in the presence of a source. They generate the full Green functions for the theory. We would like to evaluate this object in an approximation scheme where the classical equations of motion dominate, which in the scalar field case is denoted ϕ_c which satisfies

$$(\partial^2 + m^2)\phi_c + V'(\phi_c) = J. \quad (2.11)$$

The generating functional, $W[J]$, of a connected Green function $G_{conn}(x_1, \dots, x_n)$ is

$$Z[J] = \exp[iW[J]]. \quad (2.12)$$

The quantity W (quantum action) computes only the connected part of the full Z .

The classical field¹ ϕ_c

$$\phi_c(x) \equiv \frac{\delta W}{\delta J(x)} = \frac{\langle 0, \infty | \phi(x) | 0, -\infty \rangle^J}{\langle 0, \infty | 0, -\infty \rangle^J},$$

allows construction of the *effective action* Γ . When paired up with

$$J(x) = -\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)}$$

we obtain the Legendre transform

$$\Gamma[\phi_c] = W[J] - \int d^4x J(x) \phi_c(x). \quad (2.13)$$

This change in dynamical variables between J and ϕ is like the relation between the Hamiltonian and Lagrangian

$$H(x, p) = p\dot{x} - L(x, \dot{x})$$

that interchanges the dynamical variables \dot{x} and p .

The effective action is the generator of the one-particle irreducible Green functions and can be used to find quantum corrections to the classical Lagrangian through the sum of Feynman diagrams with zero-momentum external lines:

$$\Gamma[\phi_c] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \Gamma^{(n)}(x_1, \dots, x_n) d^4x_1 \cdots d^4x_n \phi_c(x_1) \cdots \phi_c(x_n). \quad (2.14)$$

¹This terminology for classical field is used because ϕ_c is an expectation value of the quantum field.

2.2 Effective Potentials for Massless Scalar Fields

The effective potential V is defined to be a function of the quantum action $\Gamma[\phi_c]$ with ϕ_c constant [66]. The position space expansion is then

$$\Gamma[\phi_c] = \int d^4x \left[-V[\phi_c] + \frac{1}{2}(\partial_\mu \phi)^2 Z(\phi) + \dots \right] \quad (2.15)$$

where Z is an ordinary function and is called a wave function renormalisation. The classical potential is replaced by effective potential for spontaneous symmetry breaking. Alternatively, one can think of the effective potential as adding quantum corrections to the classical potential. So the effective potential is the quantum action for x -independent (zero momentum) fields. It can not be known exactly since the loop expansion is very difficult to compute except for trivial examples. However, this does not mean we do not have to try. After all, the effective action for quadratic actions have a universal property of a logarithm of the fields. The one-particle irreducible Green function is [56]

$$\Gamma^{(n)}(x_1, \dots, x_n) = \int d^4k_1 \dots d^4k_n (2\pi)^{-n+1} \delta^4(k_1 + \dots + k_n) \times \exp\left(\sum_{i=1}^n k_i x_i\right) \Gamma^{(n)}(k_1 \dots k_n). \quad (2.16)$$

Note that $\Gamma^{(n)}(k_1 \dots k_n)$ and $\Gamma^{(n)}(x_1 \dots x_n)$ are related by a Fourier transform and are distinguished only by the dependence on momentum k_i or position x_i . Putting this into Equations (2.15) and (2.14) we get a formula for the effective potential

$$V(\phi_c) = - \sum_n \Gamma^{(n)}(0, \dots, 0) [\phi_c]^n, \quad (2.17)$$

where constant classical background fields have been used to eliminate the derivative terms.

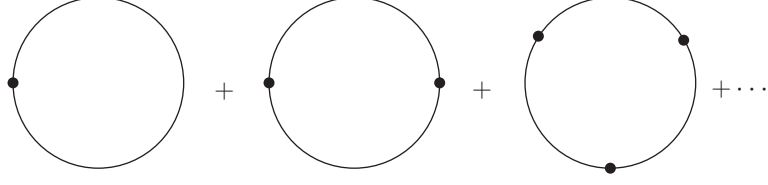


Figure 2.1: The one-loop contributions to the effective potential. The dots indicate the insertion of zero-momentum external fields appropriate for the model.

Consider the effective potential arising from the Lagrangian

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_I \\ &= \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}\phi^4\end{aligned}\tag{2.18}$$

In this ϕ^4 theory, Figure 2.1 represents the contribution of one-loop diagrams to the effective potential that would be obtained via (2.17)

$$\begin{aligned}V_1 &= i \sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left[\lambda \frac{1}{k^2 + i\epsilon} \frac{\phi^2}{2} \right]^n \\ &= -\frac{1}{2}i \int \frac{d^4k}{(2\pi)^4} \ln \left(1 - \frac{1}{2} \frac{\lambda\phi^2}{k^2 + i\epsilon} \right).\end{aligned}\tag{2.19}$$

However, this is not the method that will be used in this thesis. Instead, the effective potential will be related to the determinant of an operator which will be computed using other methods.

“This view became untenable starting in the 1970s when it was realized that there is a lot more to quantum field theory than Feynman diagrams.” [15]

2.2.1 Background Field Method in Quantum Field Theory

The background field method of computing Green functions is one way of obtaining the partition function or effective potential [70, 55]. The fields of the theory are split into background and quantum fluctuations of the fields in question. Then, for the effective

potential, the quadratic terms are kept since they give the one-loop contributions. The mass matrix is obtained in this way and shall be defined as the coefficient of the quadratic quantum fluctuations. Let the fields

$$\phi_i \rightarrow f_i(x) + h_i(x) , \quad (2.20)$$

where ϕ is an arbitrary (bosonic or fermionic) field and fluctuation h is the dynamical part. This is put into the Lagrangian, expanded to quadratic order in h , and then the functional integral is computed. This quadratic expansion can be seen to correspond to the one-loop contributions by comparison with Figure 2.1. This method is used in many aspects of quantum field theory. The generating functional for Green functions is

$$Z[f_i, J_i] = \int dh_i \exp \left(\int dx [\mathcal{L}(f_i + h_i) + J_i h_i] \right) . \quad (2.21)$$

In this way a determinant appears in the denominator for fermions and numerator for bosons. This following result is useful:

$$\int dy_1 \dots dy_n \exp(-\frac{1}{2} Y^T A Y) = (2\pi)^{n/2} (\det A)^{-1/2} , \quad (2.22)$$

where we have used Y for a vector and A is a symmetric matrix.

When both bosonic and fermion fields are involved, the end result after some work is [62]

$$Z[f_i, 0] = \text{sdet}^{-1/2} [M_{ij}(f_j)] \quad (2.23)$$

where M is the matrix also known as the quadratic form with \mathbb{R} (real) elements and sdet denotes the super-determinant which encompasses both bosonic and fermionic fields. In the next chapter we will see that these elements shall become differential operators and

the zeta function will be used for the determinant.

CHAPTER 3

DETERMINANTS, DIFFERENTIAL OPERATORS AND VECTOR BUNDLES

The mathematics necessary for discussing the functional form of gauge theories, effective potentials, effective actions and anomalies are discussed in this chapter. I leave some proofs for the references and only give complete proofs for important results.

3.1 The Geometry of Gauge Field Theories

In this section some mathematics used in aspects of gauge theories is discussed. More detail can be found in [46, 52, 53, 42, 54]. This presentation is mainly intended to make the construction more precise, up to date, and to better connect the mathematics and physics. Since this represents only the classical aspects we wait until the introduction of radiative corrections to express the quantum aspects in the form of determinants already introduced.

3.1.1 Vector Bundles, Sections, and Connections

Let M be a compact orientable Riemannian manifold. Let $\pi : \mathcal{E} \rightarrow M$ be a infinitely differentiable map, also called smooth map, from the manifold \mathcal{E} , the total space, to another manifold M , the base space. This is what is called a *fibre bundle* [22, 46, 51, 52, 53, 42, 54].

Consider on the fibre bundle one typical fibre E . Then there is a diffeomorphism

$\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times E$. A vector bundle is a fibre bundle which possesses as vector spaces as typical fibres. A one dimensional vector bundle is called a line bundle.

3.1.2 Lie Groups, Lie Algebras and Symmetry

Lie groups are an important part of any theory that may have symmetry. Lie groups are discussed in detail in [44, 45].

Let G be the symmetry group which is a Lie group and M is the base manifold. The Lie algebra is denoted by \mathfrak{g} with group multiplication, \circ . A Lie group is a group which means it satisfies the following properties

- Closure: $A, B \in G$ then $A \circ B \in G$
- Associativity: $A \in G, B \in G$ and $C \in G$ we have $(A \circ B) \circ C = A \circ (B \circ C)$
- Identity: I exists and is defined by $B \circ I = I \circ B = B$.
- Inverse: There exists an element B which gives, for $C \in G$ such that $B \circ C = I \Rightarrow C^{-1} \equiv B$

A Lie group has both a group structure along with a manifold structure. The manifold parametrizes the Lie group. So mathematically there exists a local Euclidean structure on the Lie group which means the Implicit Function theorem and Inverse Function theorem are applicable and thus calculus can be done. See Spivak [47] for details.

3.1.3 Principal Bundles

A Principal bundle P over a manifold M is itself a manifold. We have connections on P also called gauge potential A . The matrix Lie groups G acts on P to the right via

$R_g : P \rightarrow P$ or $R_gp = pg$. This g action is free meaning $g \in G$ and $R_gp = p$ implies $g = e$ the identity. Some Lie groups are important in many areas of physics.

A connection on P is a \mathfrak{g} valued 1-form ω with the properties

1. $\omega(B) = B$, $\forall B \in \mathfrak{g}$
2. $\omega_{pg}(R_{g*}X) = g^{-1}\omega_p(X)g$, $\forall X \in T_pP$ with $p \in P$ and $g \in G$ also we have a differential map $R_{g*} : T_pP \rightarrow T_{pg}P$.

There are several equivalent ways to define the connection, one of which is to view the connection as defining horizontal subspaces of the tangent space to the bundle P . In physics one usually devises a scheme where the powerful machinery of vector bundles is enlisted. The principal bundle and vector bundles are then equivalent. However, principal bundles admit a section only if it is trivial, whereas vector bundles always have them. This is not to say that sections do not exist locally; principal bundles are locally trivial.

In relating the geometry of gauge fields to physics we need locality and coordinates. This requires the bundle P to have a covering by open sets in \mathbb{R}^n to which it is locally a product, $G \times U$, where U is an open set in spacetime.

Topologically the gauge group of electroweak theory is $S^3 \times S^1$. To establish notation and for completeness, the matrix representation of the symmetry groups are

$$SU(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \text{ (complex)}, |a|^2 + |b|^2 = 1 \right\}, \quad (3.1)$$

$$U(1) = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} : \theta \in \mathbb{R} \right\}. \quad (3.2)$$

Let \mathfrak{g} be the Lie Algebra which satisfies the usual relation,

$$[T_a, T_b] = if_{abc}T_c ,$$

where the numbers f_{abc} are called the *structure constants*. Sometimes i omitted from the definition of the Lie Algebra.

The space of connections is denoted $\Omega^1(M; \mathfrak{g})$ which is a Lie algebra valued one forms. Locally the connection is given by,

$$A = A_\mu^a(x)T^a dx^\mu. \quad (3.3)$$

The two form F is an element of $\Omega^2(M; \mathfrak{g})$ known as the curvature of A . The non-abelian curvature is

$$\begin{aligned} F &= dA + A \wedge A \\ &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu, \end{aligned} \quad (3.4)$$

and the internal indices have been suppressed until further notice. The field content of electroweak theory is the connections of $\Omega^1(M, \mathbb{R}^3 \oplus \mathbb{R} \cong \mathfrak{g} = su(2) \oplus u(1))$ and the Higgs which is in $\Omega^0(M, \mathbb{C})$.

The generators of the $SU(2)$ are $\tau_i = \sigma_i/2$ ($i = 1, 2, 3$) and coupling g with connections A , while B is the connection of hypercharge symmetry $U(1)$ with coupling $g'/2$.

Usually a determinant is defined as a section of a line bundle as I'll show in the case of the zeta function. This notion of a determinant has been applied many times in physics, usually in applications to anomalies. The anomalies which are one-loop effects amount

to computing determinants as a way to see violations of symmetries at the quantum level. In the case of symmetry breaking, a field acquires a non-zero vacuum expectation value, thereby reducing the symmetry to a required result such as the $U(1)$ theory from electroweak theory.

A section s of a vector bundle is a map from the base space M to the total space \mathcal{E} such that it is compatible with the projection map π . In the physics literature sections are just the vector valued fields. However bundles which are spin bundles are associated with chiral fermions .

3.2 Differential Operators

Define the Laplacian to be $\Delta^2 = dd^* + d^*d$ where d is the exterior derivative $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ such that the condition $d^2 = 0$ is satisfied. In order to have the operator $*$, a metric has to exist on the space M [23, 25]; it is a linear elliptic partial differential operator.

Let M be a differentiable manifold and \mathcal{E} be a vector bundle. There is an algebra of differential operators on \mathcal{E} which is denoted by $\mathcal{D}(M, \mathcal{E})$

The symbol of the elliptic operator is defined by

$$\sigma_k(D)(x, \xi) = \lim_{t \rightarrow \infty} t^{-k} (e^{-itf} \cdot D \cdot e^{itf})(x) . \quad (3.5)$$

.

Let us make the identification of

$$\Gamma(M, S(TM) \otimes \text{End}(\mathcal{E}))$$

with sections inside $\Gamma(T^*M, \pi^*\text{End}(\mathcal{E}))$ that are polynomial in the fibres of the cotangent bundle T^*M . Furthermore the differential operator D of order k is *elliptic* if the section $\sigma \in \Gamma(T^*M, \pi^*\text{End}(\mathcal{E}))$ is invertible on an open set in it. This is equivalent to the coefficient matrix having all positive or negative eigenvalues. Then the symbol can be computed as

$$\sigma_k(D)(x, \xi) = \frac{(-i)^k}{k!} (\text{ad } f)^k D \quad (3.6)$$

where f is any smooth function on M and $\xi = df$. The generalized Laplacian, Δ , is a second-order differential operator such that

$$\sigma_2(\Delta)(x, \xi) = |\xi|^2 ,$$

where it is defined on a vector bundle \mathcal{E} over a Riemannian manifold M . This will appear later in the differential operator for the scalar Higgs field when I compute the one-loop effects on symmetry breaking. I present details here for completeness and to show that the results are sound mathematically.

The Laplace-Beltrami operator is,

$$\Delta = - \sum_{ij} g^{ij}(x) \partial_i \partial_j + \text{first order terms} , \quad (3.7)$$

where g is the metric on the cotangent bundle. Again g has non-zero positive eigenvalues and is therefore an example of an elliptic operator because it is invertible. In particular, for flat space the Laplacian is an elliptic operator.

3.3 Heat Kernel Constructions

Suppose we are given an elliptic operator Δ on a compact connected oriented manifold M .

Then the heat kernel $H(x, y, t) \in C^\infty(M \times M \times \mathbb{R})$ is defined as follows:

- $(\partial_t + \Delta_y)H(x, y, t) = 0, \quad t > 0,$
- $\lim_{\tau \rightarrow 0^+} \int H(x, y, \tau) f(y) d\text{vol}_y = f(x), \quad \forall f \in L^2(M)$

The scalar heat kernel, where $\Delta = \partial^2$, is given by

$$H(x, y, \tau) = \frac{1}{(4\pi\tau)^{n/2}} \exp(-(x - y)^2/4\tau) \quad \text{on } \mathbb{R}^n. \quad (3.8)$$

One can multiply by a term $\exp(-m\tau)$ for the case of a constant term in the Lagrangian which we shall need in the case of constant background field in scalar field theory (*e.g.*, ϕ^4 theory).

3.4 Riemann zeta Function

On the half plane $\text{Re}(s) > 1$ the Riemann ζ function can be defined by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \quad (3.9)$$

The function $\zeta(s)$ extends to a meromorphic function in \mathbb{C} and a order 1 pole for $s = 1$ with residue 1. A proof can be found in [5, 38, 39, 23].

For an elliptic operator Δ with eigenvalues λ_i we define the generalized zeta function

using the Mellin transform

$$\zeta_{\Delta}(s) = \sum_{\lambda_i \neq 0} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_i e^{-t\lambda_i} \right) dt. \quad (3.10)$$

Let Δ be a symmetric second order differential elliptic operator; $\Delta : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$ which are the sections of a Hermitian vector bundle \mathcal{E} .

3.5 Functional Determinants

When computing certain quantities, Green functions, in a quantum field theory the first thing that is done is approximate the quantity in a reasonable way so that the computation can be done. An important object to compute is the various Green functions which can in principle be obtained by a generating functional denoted by $Z[J]$. Many of these calculations result in functional determinants [see *e.g.*, Eq. (2.22)].

3.5.1 Determinants

In this section we review a modern definition of a determinant of a matrix A [75]. This definition has the advantage that it can be generalized to infinite dimensional vector spaces. Let A be a linear map on a n dimensional vector space V . It is an element of $End(V)$. We let A act on the highest exterior power of V , $\det V = \wedge^{max} V$. We then have an endomorphism $\det A$

$$\det A : \det V \rightarrow \det V.$$

More explicitly, pick a basis, $v_1, \dots, v_n \in V$, then

$$\det A(v_1 \wedge v_2 \wedge \dots \wedge v_n) = Av_1 \wedge \dots \wedge Av_n. \quad (3.11)$$

For matrices, such as a $n \times n$ matrix A , we recover the classical formula,

$$\det A \equiv \epsilon_{i_1 i_2 \dots i_n} a_{1i_1} a_{2i_2} \dots a_{ni_n}$$

If A is a symmetric matrix then $\det A$ is a multiplication by a complex number which is the product of eigenvalues of A .

3.5.2 Functional Determinants on Differential operators

For the example of a finite dimensional square matrix L with nonvanishing eigenvalues, we have

$$-\ln \det L = \frac{d}{ds} \left(\sum_i \lambda_i^{-s} \right) \Big|_{s=0} = \frac{d}{ds} \text{Tr}(L^{-s}) \Big|_{s=0} = \frac{d}{ds} \zeta_L(s) \Big|_{s=0}$$

which can be used to define the determinant of a differential operator. Convergence issues aside, this is enough for our purposes and resort to Seeley's theorem which makes an analytic continuation to the whole complex plane. This means that we can just define the determinant of the Laplacian the same way and then the eigenvalues λ_i can be used to express the continuation of the generalized zeta function. In actual fact it is just the generalised Riemann zeta and even further what is called in Mathematics the L -functions and automorphic forms (which occur in number theory and string theory). It is not a surprise that such object has appeared in quantum field theory since one is just working with Green functions of the Laplacian on a four sphere. So zeta functions are important for calculations in quantum field theory when no other way seems possible.

CHAPTER 4

THE STANDARD MODEL OF PARTICLES

The Standard Model (SM) is a gauge theory based on the group $SU_c(3) \times SU_L(2) \times U_Y(1)$, where the gauge fields mediate the forces that transmit energy from one particle to another. These processes are sometimes represented by Feynman Diagrams. From these diagrams one can compute the quantities of a process. The Feynman rules for these calculations can be found in virtually all books on quantum field theory. Here I shall mostly deal with tree-level and one-loop level. The one-loop effective potential has been computed using determinants of elliptic operators then extended to other fields yielding many important results, both mathematically and physically.

The Pauli matrices for a basis which span the Lie Algebra $\mathfrak{su}(2)$ are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.1)$$

Let $\tau_i = \sigma_i/2$, then the charge is then defined to be

$$Q = \tau^3 + \frac{Y}{2} = \begin{pmatrix} \frac{1}{2} + \frac{Y}{2} & 0 \\ 0 & -\frac{1}{2} + \frac{Y}{2} \end{pmatrix}, Y \in \mathbb{R}. \quad (4.2)$$

There are two different gauge field terms from the $SU(2)$ curvature ($W_{\mu\nu}$) and the $U(1)$ curvature $B_{\mu\nu}$. The kinetic term in the Lagrangian originate from these curvature

terms:

$$\mathcal{L}_{KE} = -\frac{1}{2}\text{Tr}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} , \quad (4.3)$$

where A and B are one-forms corresponding to the massless gauge fields of weak and electromagnetic interactions, and $W = dA + A \wedge A$ and $B_2 = dB^1$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \quad (4.4)$$

The case of $SU(N)$ gauge theory gives the wrong prediction of $N^2 - 1$ massless vector bosons which would propagate long range interactions contrary to the short range nature of the weak interaction that arises from the observed masses of the Z and W gauge bosons. As such, this unbroken $SU(N)$ gauge theory was not favoured as a viable theory of the weak interactions.

The gauge transformation of the connection is, for elements of a Lie group $g(x)$,

$$A_\mu(x) \rightarrow {}^g A_\mu(x) = g^{-1}(x)A_\mu(x)g(x) + g^{-1}(x)\partial_\mu g(x) . \quad (4.5)$$

This causes a redundancy in the functional integral therefore must be fixed on a gauge orbit. Quantisation also requires introduction of other non-physical fields as an artifact of gauge fixing.

In the case of $G_1 \times G_2 \times \dots \times G_n$, there will be n connections, A_i for $i = 1, \dots, n$ along with different couplings, g_i . The covariant derivative can be written as

$$D = d + g_i A_i.$$

We will only need the case of $n = 2$ for electroweak theory. This is just a parametrization

¹With an abuse of notation B_2 is an abelian 2-form.

of the inner product on the Lie algebra.

4.1 Quantising the Gauge Theories

Let us consider the generating functional, the group of gauge transformations $\mathcal{G}(P)$ and the space of connections: $\mathcal{A}(P)$,

$$Z = \int \mathcal{D}A \exp(-S[A]) , \quad (4.6)$$

where $S[A]$ is the Yang-Mills action functional (a map from gauge connections $\mathcal{A}(P)$ to \mathbb{C}) and $\mathcal{D}A$ is the measure of the \mathcal{G} -orbits and is not well defined. The gauge needs to be fixed to reduce the over-counting due to the gauge transformation in Equation (4.6). Such a fixture means choosing a gauge and we use

$$G^a(A_\mu^a) = \partial_\mu A^{\mu a} = 0. \quad (4.7)$$

The functional integral above can not be changed so insertion of a 1 will do the trick and use a representation of it as

$$1 = \int [dg] \delta(G^a(A)) \Delta[A]$$

where $\Delta[A] = \det \left(\frac{\partial G^a(A)}{\partial g} \right)$.

The quagmire of the unitary problem can be adverted by segueing into the use of ghost fields to respond to the Fadeev-Popov determinant [63]. The ghost fields are sections with opposite statistics and are a basic ingredient for the functional integral formalism of gauge theories. They are needed for a successful implementation of Feynman rules for

Yang-Mills theory.

4.2 Electroweak Theory or $SU(2)_L \times U(1)_Y$ - Bundle

β -decay was the first observed weak interaction process and was originally explained by an effective theory with four-Fermi interaction. However, this model is non-renormalisable, and later it was realized to be a effective theory arising from the spontaneously broken gauge theory.

$$SU(2)_L \times U(1)_Y \xrightarrow{\langle \phi_0 \rangle} U(1)_{em}. \quad (4.8)$$

Let $U(2)$ denote the unitary group which has four elements and is locally isomorphic to $SU(2) \times U(1)$. The bilinear metric on the groups give us the two coupling constants, which is a parametrization. The representation of the Lie group $SU(2) \times U(1)$ has four generators $(2^2 - 1)$ from $SU(2)$ and one from $U(1)$, $T^i = i\sigma^i/2$ and $Y/2$, respectively. These groups have connections and they are denoted A_μ^i for $SU(2)$ and B_μ for $U(1)$. Furthermore, they have couplings g and $g'/2$. The punchline is that the Standard Model is a weakly coupled theory in the Higgs phase which has massive gauge bosons. So it might be naive to suggest that at the topological level $S^3 \times S^1$ becomes S^1 without explaining the vacuum expectation value.

4.3 Higgs Mechanism and Spontaneous Symmetry Breaking

As noted earlier, the left-handed nature of the $SU(2)_L$ implies that naive inclusions of the fermion mass terms is forbidden by gauge invariance. The Higgs mechanism, sometimes referred to as the Brout-Englert-Higgs mechanism, Higgs-Kibble mechanism or

Anderson-Higgs mechanism, is a popular means of acquiring mass in particle physics via spontaneous symmetry breaking. Before reviewing the Higgs mechanism I will briefly discuss the relation between the range of an interaction and massive vector particles. The use of forms are important in theoretical physics so a very short introduction is included.

Differential p -forms are functions that are differentiable and assigns to each x an element of a totally antisymmetric space $\Omega^p(M)$. The symbol, $*$, is called the Hodge star operator and is defined as a map $*$: $\Omega^p \rightarrow \Omega^{4-p}$ on the four manifold M . A metric is also needed to define the Hodge star. Maxwell's equations can be derived from the field equations for F with

$$F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

The dx 's are used to form a basis of vector spaces of differential forms. The condition $dF = 0$ means there exists a local one-form $A \in \Omega^1(M)$ where the Poincaré Lemma is applied. F is a closed form. So along with the field equation for $*F$ and topological conditions for F we get the Maxwell equations for electromagnetism. When there exists a 1-form A such that $F = dA$ then F is called exact and interestingly enough not an element of De Rham cohomology $H_{DR}^2(\mathbf{R}^4)$.

Let's derive the field equations as done in Thirring [48] with variation δ_v of A by $A \rightarrow A + \delta_v A$, where v states variation rather than a coderivative

$$\delta = *d*,$$

which implies $\delta^2 = 0$. For later convenience I shall introduce the Laplace operator $\Delta = d\delta + \delta d$. Then the variation of the action is

$$\delta_v S = \int_M \delta_v A [*J + d * dA] + \int_{\partial M} \delta_v A \wedge *dA$$

the condition for boundary is $\delta_v A|_{\partial M} = 0$ which imply equations of motion for connection A

$$\Delta A = J$$

with the Lorentz gauge $\delta A = 0$. This shall be enlightening when a mass term is included. Note that Δ has a mostly positive signature and is often written as a d'Alembertian in the physics literature. However, I chose the Δ notation as it is most familiar to me.

4.3.1 Massive Gauge Potential

It seems like the only term that gives mass to the vector potential A is a term

$$\frac{1}{2}m^2 A \wedge *A , \tag{4.9}$$

where the parameter m stands for a mass. It is the only possibility due to the fact that it is the geometric term made out of A 's that is a 4-form and integrates to a real number. The $*A$ is a global 3-form and A is a global 1-form. Global means it is defined everywhere on the chart of a manifold and is only given a vector bundle when one considers monopoles (in which case is a $U(1)$ line bundle). The equation for mass can be written in local coordinates as

$$m^2 \eta_{\mu\nu} A^\mu A^\nu dx^1 \wedge \dots \wedge dx^4 .$$

To show that this term breaks $U(1)$ symmetry, apply the gauge transformation $A \rightarrow A + d\Lambda$

$$A \wedge *A \rightarrow A \wedge *A + \text{terms} \tag{4.10}$$

where terms are terms in $d\Lambda$ that do not vanish thereby breaking $U(1)$ symmetry.

The action is

$$S = -\frac{1}{2} \int_M F \wedge *F - A \wedge *J + \frac{1}{2} m^2 A \wedge *A , \quad (4.11)$$

from which we get the equation of motion for massive spin one vector fields. The resulting equation of motion is

$$(\Delta - m^2)A = J .$$

It is a local quantity and I have used the following result,

$$\delta_v \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \delta_v g_{\mu\nu} .$$

Other important identities are

$$\int \phi \wedge * \psi = \int \psi \wedge * \phi$$

and

$$\int d\phi \wedge * \psi = \int \phi \wedge * \delta \psi ,$$

where $\phi \in \Omega^{p-1}$ and $\psi \in \Omega^p$.

It is quoted early in [63] that the mass limit of the photon is $m_\gamma < 6 \times 10^{-22} \text{ MeV}/c^2$.

The equation of motion in the Lorentz gauge, $\delta A = 0$, is

$$(\Delta - m^2)A = J ,$$

which has a familiar form as that of the massive scalar field. Being massless, in this case, would mean the fields have infinite range and including a mass term puts a Yukawa-like exponential factor on the $1/r$ effect. The term is similar to e^{-mr}/r so experimentally m

is very small. It essentially vanishes. This is further seen when one uses the Modified Helmholtz equation (see page 598 of [50]) for the Green function of $\nabla^2 - m^2$ which arises for the space component of the full four-dimensional equations.

The massive photon field has a Yukawa type behaviour $\exp(-mr)/r$. To show this, start with

$$(\nabla + m^2)A = J \quad (4.12)$$

and then take the time component $A = \Phi$ to give a three dimensional Helmholtz equation

$$(\nabla^2 - m^2)\Phi = J. \quad (4.13)$$

For a system of spherical symmetry with $J = \delta^{(3)}(r)$ so we have the equation for Green function

$$(\nabla^2 + m^2)G = \delta^{(3)}(\vec{r} - \vec{r}')$$

$$G = \int d^3p \frac{\exp(ip \cdot x)}{(p^2 + m^2)}$$

and using the Partial wave expansion [49, 26] and the measure $d^3p = p^2 dp d\Omega$. There are Bessel Functions $j(z)$ and spherical harmonics $Y_l^m(\theta, \phi)$ and

$$\int_0^\pi \int_0^{2\pi} Y_l^m Y_{l'}^{m'*} d\Omega = \delta_{ll'} \delta_{mm'}$$

so

$$\int p^2 dp \frac{j_0(pr)}{(p^2 + m^2)} = \int p^2 dp \frac{\sin(pr)}{pr(p^2 + m^2)}$$

Equation 3.723(3) of [26] with $\alpha \rightarrow r$, $x \rightarrow p$ and $\beta \rightarrow m$ we have the integral

$$\frac{1}{r} \int_0^\infty \frac{x \sin(\alpha x) dx}{(x^2 + m^2)} = \frac{\pi}{2r} e^{-mr} \quad (4.14)$$

so it is shown that the static field of a massive photon field decays exponentially. Therefore any massive vector particle has this property. However, such terms have the very unwanted breaking of gauge invariance.

4.4 Electroweak Symmetry Breaking

Long-range forces are propagated by connections on a vector bundle and when a mass term is used we get a short-ranged interaction such as that in weak interactions. However, when such a term is added to the Lagrangian it induces a non-renormalisable interaction which makes quantities incalculable. So another method is needed to generate massive connections and hence the short-range nature of the weak interaction. This is a segue into spontaneous symmetry breaking.

4.4.1 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is losing generators of a representation of a Lie group which acts on states of the theory. A symmetry is spontaneously broken if a residual generator, say G , acting on the vacuum fails to vanish ; $G|0\rangle \neq 0$. As discussed below this will mean that the residual symmetry of quantum electrodynamics $U(1)$ has survived spontaneous symmetry breaking via the Higgs mechanism (see, *e.g.*, [56]). We obtain a massive vector theory that is renormalisable [31]. From a mathematical perspective spontaneous symmetry breaking is basically a bundle reduction [53]. The Higgs field is a $Y = 1$ complex doublet of $SU(2)$ and denoted as

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{4.15}$$

so that $\Phi \in \mathbb{C}^2$. Being complex there will be four degrees of freedom in Φ . The Lagrangian is

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi), \quad D_\mu = \partial_\mu - igA_\mu^a \tau^a - i\frac{g'}{2}B_\mu. \quad (4.16)$$

The following potential is $SU(2)_L$ invariant and renormalisable, with the interesting case $\mu^2 > 0$,

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \quad (4.17)$$

At this point in the discussion the parameter μ is not interpreted as a mass because of an explicit negative sign. We will see that after spontaneous symmetry breaking will be interpreted as a mass of a scalar particle. We need to find a non-vanishing minimum of the potential which is a condition defined by

$$\frac{\partial V(\Phi)}{\partial \Phi} = 0. \quad (4.18)$$

If the parameters satisfy the conditions

$$\begin{aligned} \mu^2 &> 0, \\ \lambda &> 0, \end{aligned} \quad (4.19)$$

the solutions to (4.17),

$$-2\lambda \Phi^\dagger \Phi + \mu^2 = 0, \quad (4.20)$$

exist and are non-trivial. First let us change the above into real fields

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ v + i\phi_4 \end{pmatrix} \quad (4.21)$$

thereby writing

$$\Phi^\dagger \Phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + v^2 + \phi_4^2).$$

The quantities, μ and λ , are parameters and not a mass as we have a negative term. From the invariance of the potential under gauge transformation we can choose to set some of the ϕ 's to zero, say $\phi_1 = \phi_2 = \phi_4 = 0$. The minimum of the potential is

$$\Phi^\dagger \Phi = \frac{1}{2}v^2 = \frac{\mu^2}{2\lambda}.$$

In the minimum energy configuration the vacuum expectation value for the state Φ is not zero and can be chosen to be an arbitrary point with a rotation in isospin space. The vacuum expectation value for the Higgs is *chosen* to be,

$$\Phi_0 = \langle 0 | \Phi | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (4.22)$$

This implies that $v^2 = \mu^2/\lambda$.

Acting on this vacuum expectation value by Q we have

$$Q\Phi_0 = \begin{pmatrix} \frac{1}{2} + \frac{Y}{2} & 0 \\ 0 & -\frac{1}{2} + \frac{Y}{2} \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad (4.23)$$

and this vanishes only in the case $Y_\phi = 1$ as a definition of hypercharge of the Higgs. The electric charge of this Higgs is zero. So the vacuum defined by Φ_0 preserves the generator Q so the connection A_μ remains massless and unbroken. This is the origin of the $U(1)$

gauge theory

$$T_3 \Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0. \quad (4.24)$$

This means that the $SU(2)_L$ symmetry is spontaneously broken by the vacuum expectation value.

Expanding the field around this vacuum expectation value Φ_0 we get

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix}. \quad (4.25)$$

and then putting this into the Lagrangian we can get masses of particles. This form of the field is needed to see the effects of the vacuum expectation value on field masses.

The gauge boson masses are obtained from the kinetic term, ignoring the ∂_μ part,

$$\begin{aligned} (D_\mu \Phi)^\dagger (D^\mu \Phi) &\Rightarrow \frac{1}{2} (0 \ v) (g\tau^i A_\mu^i + \frac{1}{2} g' B_\mu)^2 \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{2} (0 \ v) \frac{1}{4} \begin{pmatrix} g'B + gA^3 & g(A^1 - iA^2) \\ g(A^1 + iA^2) & g'B - gA^3 \end{pmatrix}^2 \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{2} \frac{v^2}{4} \{ g^2 [(A_\mu^1)^2 + (A_\mu^2)^2] + (gA_\mu^3 - g'B_\mu)^2 \} \\ &= \frac{1}{2} v^2 \left\{ \frac{1}{2} g^2 \left(\frac{1}{\sqrt{2}} \right)^2 (A_\mu^1 - iA_\mu^2)(A_\mu^1 + iA_\mu^2) + \frac{1}{4} (g^2 + g'^2) \left[\frac{gA^3 - g'B}{\sqrt{g^2 + g'^2}} \right]^2 \right\} \\ &= \frac{1}{2} v^2 \left\{ \frac{1}{2} g^2 W^+ W^- + \frac{1}{4} (g^2 + g'^2) (Z^\mu Z_\mu)^2 \right\}. \end{aligned} \quad (4.26)$$

The factor in front of the second term makes the couplings combine rather than have two

separate fields which are related by couplings. I have removed the index μ for clarity of presentation. This shows that the presence of a non-zero vacuum expectation value produces mass terms in an otherwise massless Lagrangian.

Notice that the symmetry breaking mechanism ensures that the photon remains massless. If it were massive the universe would be a very different place. Instead of the Maxwell's equations we would have the Proca equation! In this case we might then end up with a term

$$\frac{1}{2}m^2 A^2$$

for a massive photon and all the generators of the $SU(2) \times U(1)$ are broken. Surely something stopped symmetry breaking at $U(1)_{em}$ not the anthropic principle. The charge generator acting on this version of Φ_0 is nonzero so that again $U(1)$ is completely broken as well.

The following interpretation of the representations of the massless connections as differential forms was used (4.26),

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2) && \text{Charged} \\ Z_\mu^0 &= \frac{gA_\mu^3 - g'B_\mu}{\sqrt{g^2 + g'^2}} && \text{Neutral} \\ A_\mu &= \frac{g'A_\mu^3 + gB_\mu}{\sqrt{g^2 + g'^2}} && \text{Neutral.} \end{aligned} \tag{4.27}$$

Note that these are elements of $\Omega^1(M, \mathbb{R}^3)$. What transpired is that we began with a principal bundle and added the Higgs field with a minimum of the potential and ended up with a set of 1-forms called W 's and Z 's.

The electroweak gauge boson have the following masses:

$$M_W = g\frac{v}{2}, \quad M_Z = \sqrt{g^2 + g'^2}\frac{v}{2} \text{ and } M_A = 0. \tag{4.28}$$

The comparison with four-Fermi theory can be used to obtain an experimental value for the vacuum expectation value v ,

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{1}{2v^2} \quad (4.29)$$

so using the high precision value of G_F we obtain

$$v = 247 \text{ GeV}.$$

It behooves us to define a rotation angle

$$\cos(\theta_w) = \frac{g}{\sqrt{g^2 + g'^2}} \text{ and } \sin(\theta_w) = \frac{g'}{\sqrt{g^2 + g'^2}}$$

which is a tree-level result expression for *weak angle* or *Weinberg angle*. According to experiments [64] we have

$$\sin \theta_w = 0.23120(15). \quad (4.30)$$

The measured values of the vector boson masses are $M_W = 80.425(38) \text{ GeV}/c^2$ and $M_Z = 91.1876(21) \text{ GeV}/c^2$ and we have a ratio M_W/M_Z that is consistent with an independently measured weak angle

$$\frac{M_W}{M_Z} = \cos \theta_W.$$

The quantity

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} \quad (4.31)$$

is an important parameter which is identity or very close to it at tree level. It depends on the vacuum expectation value of the Higgs, v . There is an expression for this in more

than one Higgs doublet and even generally for triplet. These are some of the reasons that the Standard Model is considered successful, yet there are problems with it such as the hierarchy problem and its lack of gravitational interaction which is needed in a realistic theory.

I have just described the minimal Higgs sector needed for electroweak symmetry breaking. There are other methods that explore mass generation without a scalar particle being introduced into the theory. We discuss cases, such as the Minimal Supersymmetric Standard Model (MSSM) which require a non-minimal Higgs sector.

4.4.2 Chiral Fermions, Yukawa Interactions and Mass Generation

The fact that the spacetime contains fermions means that spacetime has a spin structure. The fibre bundle of the previous sections have to be changed to accommodate the spinors. The spinors are sections of a spin bundle and the Dirac operators are maps from the sections whose quadratic form determines the elliptic operator such as the Laplacian in space-time.

As we consider particles such as electrons we need the spacetime to have structure to accommodate such objects. These are spinors. They have different statistics than bosons. Since the electroweak theory is parity violating, we need to have only left-handed spinors, which in 4-spinor notation means $1 - \gamma^5$ projections of the fields. This makes the two-spinor ideal and useful in supersymmetry.

The masses of quarks and leptons can be obtained in the same way as for bosons. The mass terms must not be the usual $m\bar{\phi}\phi$ -like term as this involves combinations $\bar{Q}_L q_r$ which break gauge invariance.

As outlined in (1.4), the gauge invariant Yukawa interaction term is

$$\mathcal{L}_{Yukawa} = y_{\alpha\beta}^d \bar{Q}_\alpha d_\beta \Phi + y_{\alpha\beta}^L \bar{L}_\alpha e_\beta \Phi + y_{\alpha\beta}^u \bar{Q}_\alpha u_\beta \Phi_c + h.c., \quad (4.32)$$

where y is the Yukawa couplings depending on the number of generations: a 3×3 matrix in the typical case.

To get the mass of say the electron, we take $y_{\alpha\beta}^L \bar{L}_\alpha e_\beta \Phi$ and put in the values of broken symmetric fields and get

$$Y_{\alpha\beta}^L \bar{L}_\alpha e_\beta \Phi \implies y^e \begin{pmatrix} \bar{\nu} & \bar{e}_L \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} e_R = \frac{y^e v}{\sqrt{2}} \bar{e}_L e_R \quad (4.33)$$

and hence gives us

$$m_e = \frac{y_e v}{\sqrt{2}}. \quad (4.34)$$

The same things happen with quarks and other leptons. The values of the Yukawa couplings are not calculable in this Standard Model. Only experiments can determine the values. For some reason y^t is almost like 1, giving the top quark mass $m_t = 247 \text{ GeV}/\sqrt{2} \sim 174 \text{ GeV}$.

Now I show how the mass of the Higgs is obtained using the classical potential, (4.17), for the complex doublet Φ . The μ^2 term changes sign from a tachyon-type particle to a massive one. Making use of Equation (4.25), the classical potential (4.17) with $\Phi^\dagger \Phi = \frac{1}{2}[v + \eta(x)]^2$ and $v^2 = \mu^2/\lambda$ becomes

$$V(\Phi) = -\frac{1}{2}\mu^2[v + \eta(x)]^2 + \frac{1}{4}\lambda[v + \eta(x)]^4. \quad (4.35)$$

Collecting terms in η^2 we get

$$-\frac{1}{2}\mu^2\eta^2 + \frac{3}{2}\lambda v^2\eta^2 = +\mu^2\eta^2 ,$$

which gives us

$$m_\eta^2 = 2\mu^2. \tag{4.36}$$

What has happened is the act of symmetry breaking: the scalar field Φ has given us a positive mass squared η particle called the Higgs. This mass is an unknown parameter and is interpreted as a mass of a scalar field η .

This classical analysis can be extended to the quantum one-loop result by calculating the effective potential [56]. The effective potential can be obtained using many different methods but I have chosen to calculate the effective potential as the determinant of certain differential operators (*e.g.*, the Laplacian). In this method the elliptic operators (which can be interpreted as Hermitian) can be diagonalized and can be taken as a product of the eigenvalues. I will show, as Coleman-Weinberg did [27], that spontaneous symmetry breaking can occur as a purely quantum effect even if $\mu = 0$.

4.4.3 Two Higgs Doublet Models

The particle content of these models is based on the Standard Model of $SU(3) \times SU(2) \times U(1)$ with minimal amount of superpartner and supersymmetry (SUSY) being broken at for example the Large Hadron Collider (LHC) scale.

Let us first parameterize the two Higgs Lagrangian with the notation of Gunion and Haber [36] (see also [34,37]) but using the notation H_1 and H_2 for the two Higgs doublets

of the weak interaction. We have the gauge invariant scalar potential

$$\begin{aligned}
V = & m_{11}^2 H_1^\dagger H_1 + m_{22}^2 H_2^\dagger H_2 - [m_{12}^2 H_1^\dagger H_2 + \text{h.c.}] + \frac{1}{2} \lambda_1 (H_1^\dagger H_1)^2 + \frac{1}{2} \lambda_2 (H_2^\dagger H_2)^2 \\
& + \lambda_3 (H_1^\dagger H_1)(H_2^\dagger H_2) + \lambda_4 (H_1^\dagger H_2)(H_2^\dagger H_1) + \left(\frac{1}{2} \lambda_5 (H_1^\dagger H_2)^2 \right. \\
& \left. + (\lambda_6 (H_1^\dagger H_1) + \lambda_7 (H_2^\dagger H_2)) H_1^\dagger H_2 + \text{h.c.} \right)
\end{aligned} \tag{4.37}$$

We take all parameters to be elements of \mathbb{R} and CP-conserving for simplicity. In a supersymmetric model, these parameters take the values

$$\lambda_1 = \lambda_2 = -\lambda_{345} = \frac{1}{4}(g^2 + g'^2), \quad \lambda_4 = -\frac{1}{2}g^2, \quad \lambda_5 = \lambda_6 = \lambda_7 = 0, \tag{4.38}$$

where $\lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5$. If one loop corrections are included along with tree level mass vanishing we simply apply these values.

CHAPTER 5

RADIATIVE ELECTROWEAK SYMMETRY BREAK- ING

5.1 Coleman-Weinberg via zeta Function Regularisation

There are perhaps several ways to get this effective potential for a $\lambda\phi^4$ theory corresponding to the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4. \quad (5.1)$$

Sometimes $V(\phi)$ will be used and ϕ^4 theory is an example. The Coleman-Weinberg effective potential can be derived in many ways. The one that I shall employ is the heat kernel technique [29] and it lends itself much more easily to Dirac operators and other operators. This is a symmetric second-order elliptic differential operator $\nabla : \Gamma(\mathcal{E}) \rightarrow \Gamma[\mathcal{E}]$ which act on sections of a vector bundle \mathcal{E} .

To obtain a result for determinants of Laplacians on Minkowski spaces, we need to find a way to regularise the infinities. The ζ -function [67, 69, 73, 74] is a simple one so I chose it. Consider an operator A which is a Hermitian, positive, semidefinite operator with eigenvalues a_n and complete and orthonormal eigenfunctions $f_n(x)$ such that

$$Af_n(x) = a_nf_n(x). \quad (5.2)$$

Let

$$\zeta_A(s) = \sum_n \frac{1}{a_n^s} \quad (5.3)$$

be the zeta function for A . For interesting further information see [30]. This now is in a four dimensional Euclidean space as is usually done in computing an integration in Minkowski space via Wick's rotation.

Note also that

$$\left. \frac{d\zeta_A(s)}{ds} \right|_{s=0} = - \sum_n \ln(a_n) a_n^{-s} \Big|_{s=0} = - \ln \prod_n a_n . \quad (5.4)$$

This implies that the determinant of the operator A is

$$\det A = \prod_n a_n = e^{-\zeta'_A(0)} = e^{\text{Tr}(\ln A)} , \quad (5.5)$$

which I shall use to compute the effective potential of an operator of the connected functional integral.

5.1.1 Loop expansion of the Effective Potential

Spontaneous symmetry breaking makes use of the effective potential and amounts to quantum corrections. Thus we need to evaluate this [65, 66, 70] beginning with the definition

$$\exp iW = \exp \left(\frac{i}{\hbar} S[\phi_0] \right) \{ \det[\partial^2 + V''(\phi_c)] \}^{-\frac{1}{2}} \quad (5.6)$$

and only holds for $J = 0$. First one begins with a path integral

$$\exp iW[J] = \int_{\mathcal{M}} \mathcal{D}\phi e^{(i/\hbar)S[\phi, J]} \quad (5.7)$$

where \mathcal{M} is the space of fields over which we integrate and

$$S[\phi, J] = \int d^4x [\mathcal{L}(\phi) + \hbar \phi(x) J(x)]$$

where when one assumes the ϕ^4 theory. There is also an assumption of compact support for this to be an Riemann integral [46]. The first variation is

$$\left. \frac{\delta S}{\delta \phi} \right|_{\phi_0} = \hbar J(x) , \quad (5.8)$$

which will be used in the following second-order expansion around ϕ_0 to obtain:

$$\begin{aligned} S[\phi, J] &= S[\phi_0, J] + \int dx [\phi(x) - \phi_0] \left. \frac{\delta S}{\delta \phi(x)} \right|_{\phi_0} \\ &+ \int dx dy [\phi(x) - \phi_0] [\phi(y) - \phi_0] \left. \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \right|_{\phi_0} + \cdots \\ &= S[\phi_0] + \hbar \int dx \phi(x) J(x) \\ &+ \frac{1}{2} \int dx dy [\phi(x) - \phi_0] \left. \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \right|_{\phi_0} [\phi(y) - \phi_0] + \cdots \end{aligned} \quad (5.9)$$

and the definition for the second-order variation is

$$\left. \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \right|_{\phi_0} = [\partial^2 + V''(\phi_0)] \delta(x - y).$$

Now set $\phi' = \phi - \phi_0$, to obtain

$$\begin{aligned} S[\phi, J] &= S[\phi_0, J] + \hbar \int dx \phi'(x) J(x) \\ &+ \frac{1}{2} \int dx \phi'(x) \left[\partial^2 + V''(\phi_0) \right] \phi'(x) + \cdots \end{aligned} \quad (5.10)$$

When this is put into Equation (5.7) to get Equation (5.6). To get the determinant we use Equation (2.22) and some functional analysis for elliptic operators. Then we have, instead of equation (5.6),

$$\exp \frac{i}{\hbar} W = \exp \left(\frac{i}{\hbar} S[\phi_0, J = 0] \right) \{ \det[\partial^2 + V''(\phi_c)] \}^{-\frac{1}{2}} \quad (5.11)$$

which holds for $J = 0$. I have derived this using the background field method for a scalar field in the potential V which has the typical polynomial Lagrangian. The calculation of the determinant requires some standard integrals:

$$\Gamma(s) \equiv \int_0^\infty t^{s-1} e^{-t} dt = \lambda^s \int_0^\infty t^{s-1} e^{-\lambda t} dt \quad (5.12)$$

and

$$U(a) \int d^4x = V(a) \int d^4x - \frac{i}{2} \text{Tr}[\partial^2 + V''(a)] . \quad (5.13)$$

Here $\hbar = 1$ and the volume integral is infinite in \mathbb{R}^4 so we work on a 4-sphere. I am anticipating a cancellation once the zeta function is computed.

The heat kernel H (sometimes $k(\tau, x, y)$) is defined for an elliptic operator Δ_x which could contain the Laplacian as

$$\Delta_x H(x, y, \tau) = -\frac{\partial}{\partial \tau} H(x, y, \tau) \quad (5.14)$$

The ζ -function can be expressed in terms of H by the *Mellin transformation*,

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \int H(x, x, \tau) dx. \quad (5.15)$$

This is the most important formula for studying aspects of obtaining the effective potential.

Note that the arguments are in the $x = y$ coincident limit so the space integral is not involved. The problem of finding the determinant lies in this part of the calculation. Once the heat equation is solved we can then just insert the heat kernel and without actually doing the complicated integral, we should simply take a derivative with respect to s at the value $s = 0$ to obtain the determinant. It should be noted that this heat kernel has a notion of dimensionality of spacetime that is not apparent in the notation. This will be shown later in the basic example.

There is an important property of H in the limit $\tau \rightarrow 0$:

$$\lim_{\tau \rightarrow 0} \int H(x, y, \tau) f(y) d^4x = f(x) , \quad \forall f \in L^2(\mathbb{R}^4) , \quad (5.16)$$

which implies $H(x, y, 0) = \delta(x - y)$. The pointwise convergence also makes it possible to write

$$H(x, y, \tau) = \sum_n e^{-a_n \tau} f_n(x) \otimes f_n^*(y) , \quad (5.17)$$

where $\{f_i\}$ are an orthonormal basis of $L^2(\mathbb{R}^4)$ and again satisfy $\Delta f_i = \lambda_i f_i$.¹ The function being pointwise convergent requires the Sobolev Embedding theorem and Fundamental Elliptic Estimates.

To compute $\det(A)$ we find solutions to heat Equation with a Δ as the boundary condition, insert the result into the definition of ζ_A , and calculate the determinant

$$\det(A) = e^{-\zeta'_A(0)} . \quad (5.18)$$

.

The symmetric elliptic operator we need to use this on is contains ϕ_c and is defined

¹Note that we are anticipating an Euclidean-space operator A (with eigenvalues a_n) that contains the Laplacian Δ (with eigenvalues λ_n).

above:

$$A_X = -\partial^2 + m^2 + \frac{\lambda}{2}\phi_c^2. \quad (5.19)$$

We would then extend the result (5.4) to include the mass parameter and $\lambda\phi_c^2$ term:

$$[-\partial^2 + m^2 + \frac{\lambda}{2}\phi_c^2]H(x, y, \tau) = \frac{\partial}{\partial\tau}H(x, y, \tau). \quad (5.20)$$

Here the extra term is just the trivial Hessian which is needed in the next section for more than one scalar field.

Returning now to the effective potential,

$$\Gamma_E[\phi_{cl}] = \Gamma_E^{(0)}[\phi_{cl}] + \hbar\Gamma_E^{(1)}[\phi_{cl}] + \cdots, \quad (5.21)$$

$$\Gamma_E^{(1)}[\phi_{cl}] = -\frac{1}{2}\zeta'_{[-\partial^2+m^2+\frac{\lambda}{2}\phi_c^2]}(0).$$

Pick $\phi_{cl} = v$ a constant field when $J \rightarrow 0$ so that the effective action is $\Gamma[\phi_c = v] = + \int d^4x V(v)$. As this is now just $\partial^2 + \text{constant}$ we take a function of τ and clearly from integration we get

$$H(x, y, \tau) = \frac{b^4}{16\pi^2\tau^2} \exp\left(-b^2\frac{(x-y)^2}{4\tau}\right) \exp\left(-(m^2 + \frac{1}{2}\lambda v^2)\frac{\tau}{b^2}\right), \quad (5.22)$$

which will satisfy the heat equation for H . I will keep the perhaps a notationally confusing arbitrary constant b here as it will cease to exist in the formulas to come. Note that $d = 4$ in this case. The trace of the heat kernel amounts to setting $x = y$.

After a change of variable we get a form that is easily solved:

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \int d^4x \frac{b^4}{16\pi^2 \tau^2} \exp\left(-\left(m^2 + \frac{1}{2}\lambda v^2\right) \frac{\tau}{b^2}\right) \\
&= \frac{b^4}{16\pi^2} \left(\frac{m^2 + \frac{1}{2}\lambda v^2}{b^2}\right)^{2-s} \frac{\Gamma(s-2)}{\Gamma(s)} \int d^4x \\
&= V b^{-s} \frac{m^D}{(4\pi)^{D/2}} \frac{\Gamma(s-D/2)}{\Gamma(s)} \left(\frac{m^2 + \frac{1}{2}\lambda v^2}{b^2}\right)^{-s},
\end{aligned} \tag{5.23}$$

where we use

$$\frac{\Gamma(s-2)}{\Gamma(s)} = \frac{1}{(s-2)(s-1)}$$

combined with (5.12). I have kept the spacetime dimension as D in the last line in (5.23) to illustrate that this is in fact a general result. The modified Bessel function K_ν arises from this scenario

$$\int_0^\infty \tau^{s-1} H(x, y; \tau) = \frac{2}{(4\pi)^{D/2}} \left(\frac{R}{2m}\right)^{s-D/2} K_{D/2-s}(mR).$$

The form of the integrand would work with any form of constant background. It has been used for a family of operators where the background is non-constant and in many instances in the heat kernel proof of index theorems. This means that the determinant line bundle is non-trivial and the anomalies are an obstruction to its triviality. In the Coleman-Weinberg potential/effective action the bundle is free of anomalies and therefore trivial and isomorphic to a open subset of a rank-1 bundle.

Putting the results together the value of the effective potential is²

$$\begin{aligned} V(v) &= -\frac{b^4}{32\pi^2} \frac{d}{ds} \left\{ \frac{1}{(s-2)(s-1)} \left(\frac{m^2 + \frac{1}{2}\lambda v^2}{b^2} \right)^{2-s} \right\} \Big|_{s=0} \\ &= \frac{1}{64\pi^2} \left[m^2 + \frac{1}{2}\lambda v^2 \right]^2 \left(-\frac{3}{2} + \ln \frac{[m^2 + \frac{1}{2}\lambda v^2]}{b^2} \right). \end{aligned} \quad (5.24)$$

Using the ϕ_c at tree level as just the original potential for $\phi = \phi_c$ and above as the loop corrections with ζ -function regularised, we have,

$$V[\phi_c] = \frac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 + \hbar \frac{1}{64\pi^2} \left[m^2 + \frac{1}{2}\lambda\phi_c^2 \right]^2 \left(-\frac{3}{2} + \ln \frac{[m^2 + \frac{1}{2}\lambda\phi_c^2]}{b^2} \right) + \mathcal{O}(\hbar^2). \quad (5.25)$$

This is seen to depend on an arbitrary renormalisable scale b^2 , which we shall remove, as well as the traditional m^2 and λ . The factor $3/2$ is here for scalars and fermions [64].

For comparison with [27], Eq. (5.25) must be converted to a massless theory ($m^2 = 0$) in the Coleman-Weinberg renormalisation scheme, where [63]

$$\lambda_{cw}(M) = \frac{d^4}{d\phi^4} V \text{ at } \phi = M,$$

and this implies

$$\ln\left(\frac{\lambda M^2}{2b^2}\right) = -\frac{8}{3} \text{ at } m^2 = 0 \quad (5.26)$$

or $\ln 2b^2 = \ln(\lambda M^2) + 8/3$. However, λ and M are related so that the condition for V stays unaffected by changes in them.

Finally we get the result of Coleman-Weinberg using the heat kernel method (5.24),

$$V = \frac{1}{4}\lambda_{cw}\phi_c^4 + \frac{\lambda_{cw}^2}{(16\pi)^2}\phi_c^4 \left(\ln \frac{\phi_c^2}{M^2} - \frac{25}{6} \right). \quad (5.27)$$

²Note that we will later change the renormalisation scale, and a $m^2 \rightarrow 0$ limit exists.

What has happened is that first the zeta function regularisation was used to define the undefined \det . This could be done generally in curved space as well and using the heat kernel expansion could have used $d^2 + m^2$ as the operator. The only thing that is tough is the use of a constant $\phi_{cl} = v$ and was put in after change of name. Also the effective action for constant ϕ just shifted the mass parameter. The volume of space being infinite was cancelled in this way. The Mellin transform was used to evaluate the ζ function and equate it to the determinant. I will try to use this technique to compute the two Higgs potential because it works quite nicely for various fields.

The Coleman-Weinberg (CW) theory [27,28] explains the origin of spontaneous symmetry breaking without introducing a Higgs mass parameter μ which can obviate some aspects of the hierarchy problem. The origin of spontaneous symmetry breaking in the CW theory is quantum corrections and hence the symmetry-breaking mechanism is known as the radiative electroweak symmetry breaking. Although the original Coleman-Weinberg scenario based upon small coupling solution $\lambda \sim e^2$ lead to an $\mathcal{O}(10\text{GeV})$ Higgs mass, which has been ruled out experimentally [34], large λ coupling solutions have been discovered which lead to a $\mathcal{O}(220\text{GeV})$ Higgs mass [40] more consistent with current experimental mass bounds.

5.2 Two Scalar Fields with the Identical Method

In this section a simple generalization of the ϕ^4 theory to the case of two scalar fields with interesting interactions will be studied in the background field method. The introduction of not one but two other couplings to the simple case poses interesting problems, which is helpful for other cases of radiative symmetry breaking of electroweak interaction that will generate a mass. Even the supersymmetric model is seen to benefit from this simple

analysis.

The introduction of interaction terms with extra scalars with same couplings are not as interesting as those with differing ones.

5.3 A Simple not-so-Simple Model

The paper [62] is the main source for definition of determinants. Consider now a compact Euclidean signature spacetime, M . On this there is a trivial bundle with two sections ϕ_i , $i = 1, 2$ with dynamics given by properties of the Lagrangian,

$$\mathcal{L}(\phi_1, \phi_2) = \sum_i (\partial_i \phi_i)^2 + \sum_i m_i^2 \phi_i^2 - \sum_i \frac{\lambda_i}{4!} \phi_i^4 + \frac{\lambda_3}{4} (\phi_1 \phi_2)^2 .$$

We would like to compute $Z[J_i]$ of this Lagrangian with two sources of fields ϕ_i . It is enough to expand up to quadratic terms in fields to get a one-loop result [62]. The effective potential can then be obtained quite readily and is an extension of ϕ^4 theory. The heat kernel is used afterwards as well as the zeta-function.

Take $\phi_i(x) = \varphi_i + h_i(x)$ ³ to obtain the usual expansions,

$$\phi_i^2(x) = [\varphi_i + h_i(x)]^2 = \varphi_i^2 + 2\varphi_i h_i(x) + h_i^2(x),$$

$$\phi_i^4 = \varphi_i^4 + 4\varphi_i^3 h_i + 6\varphi_i^2 h_i^2 + 4\varphi_i h_i^3 + h_i^4,$$

$$\phi_1 \phi_2 = \varphi_1 \varphi_2 + \varphi_1 h_2 + h_1 \varphi_2 + h_1 h_2 ,$$

$$(\phi_1 \phi_2)^2 = \varphi_1^2 \varphi_2^2 + h_1^2 \varphi_2^2 + \dots ,$$

where I have only included the quadratic terms in the last expansion. The quadratic func-

³Sometimes φ will be replaced by v to conform with other literature when it is constant.

tions are ubiquitous in most of math and M-theory and a given special status as they seem to be the only computable partition functions. Quadratic functions encode the underlying structures such as spinors and give us the Green function for many if not all of quantum field theory. Atiyah-Singer index theorem is proved with such functions for example, and the list goes on.

Combining the above terms

$$h^T A_{int} h = 6 \frac{\lambda_i}{4!} \varphi_i^2 h_i^2 + \frac{\lambda_3}{4} (\varphi_1^2 h_2^2 + h_1^2 \varphi_2^2)$$

which is the term included in the mass matrix for the one-loop effective potential. The one corresponding for quantum fluctuations of h_1 is

$$\left(\frac{\lambda_1}{4!} \varphi_1^2 + \frac{\lambda_3}{4} \varphi_2^2 \right) h_1^2.$$

We must combine the quadratic terms in h_i which can then be used to compute the effective potential for a trivial two scalar fields with a $h_1 h_2$ -type interaction. The Feynman diagram for this term is not studied here.

$$Z[J_i] = \int \mathcal{D}h_1 \mathcal{D}h_2 \exp L(h_i) , \quad (5.28)$$

$$Z[J_i] = \int \mathcal{D}h_1 \mathcal{D}h_2 \exp L(h_1, h_2) = \exp \frac{1}{2} \int h_1(x) (p^2 - m^2 - \frac{\lambda_1 \varphi_1^2}{2}) h_1(x) , \quad (5.29)$$

$$\Gamma = \frac{i}{2} \left(\ln \det [\partial_1^2 + \frac{\lambda_1}{2} \varphi_1^2 + \frac{1}{2} \varphi_2^2] \right) . \quad (5.30)$$

Consider the simple definition of a finite determinant that gives us an area of a quadratic form of two variables, x, y ,

$$\int \exp(-ax^2 - cy^2 + bxy) dx dy = \det A, \quad (5.31)$$

where

$$A = \begin{pmatrix} -a & b \\ b & -c \end{pmatrix}$$

is a matrix of the quadratic function. Choose $f = x - by/a$ and $y = y$, then we have

$$-af^2 - \left(\frac{b^2}{a} + c\right)y^2$$

and

$$A = \begin{pmatrix} -a & 0 \\ 0 & \frac{b^2}{a} + c \end{pmatrix}.$$

For two fields like the simple 2 scalars this will give us a clue into the computation of determinants, the one-loop effects. We would like to see how a similar procedure works in two scalar fields previewing more complicated two-Higgs doublet models. But let us first diagonalize it. This will depend on the sign for each of the quantities

$$A_x = \begin{pmatrix} -\partial^2 + m_1^2 & 0 \\ 0 & -\partial^2 + m_2^2 \end{pmatrix}. \quad (5.32)$$

The matrix for the two scalar fields has the quadratic matrix which is easily diagonalized through a change of variables. A diagonal matrix remains diagonal under this same change

of variables. The mass matrix is

$$A_{int} = \begin{pmatrix} \frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2 & \lambda_3\varphi_1\varphi_2 \\ \lambda_3\varphi_1\varphi_2 & \frac{\lambda_2}{4}\varphi_2^2 + \lambda_3\varphi_1^2 \end{pmatrix} \quad (5.33)$$

in

$$\int [dh_1][dh_2] \exp \left(- \int d^4x h^T A_{int} h \right). \quad (5.34)$$

This matrix may be diagonalized using the Lagrange method and this means redefining the fields again but this is not too bothersome. The field redefinition makes obtaining the zeta functions much simpler and more elegant. Once the computation is done, the cross terms are irrelevant. Let

$$g = h_1 - \frac{\lambda_3\varphi_1\varphi_2}{\frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2} h_2, \quad h_2 = h_2$$

When this is done the mass matrix is diagonal and the determinants are more readily usable. The technique of zeta function regularisation for two functions f_1 and h_1 for our new scalar fields can be applied. We compute the zeta functions for these and it is easy to see what happens. Being a new basis the two fields simply add their effects.

We now have two operators appearing on the diagonal of the matrix

$$A_{int} = \begin{pmatrix} \frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2 & 0 \\ 0 & \frac{(\frac{\lambda_2}{4}\varphi_2^2 + \lambda_3\varphi_1^2)^2}{\frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2} + (\lambda_3\varphi_1\varphi_2) \end{pmatrix}, \quad (5.35)$$

so the diagonalized form can then be used to get the two scalar fields with this form.

Let $\phi_2 = \phi_1 \tan \beta$, $A = (\frac{\lambda_1}{4} + \lambda_3 \tan \beta) \phi_1^2$, $C = (\frac{\lambda_2}{4} \tan \beta + \lambda_3) \phi_1^2$ and $B = \lambda_2 \phi_1^2 \tan \beta$.

Since the Lagrangian has been diagonalized things are a little easier and require only two heat kernels G_i corresponding the Laplacians

The heat kernel for the second operator, denoted B , has the form

$$H_2(x, y, \tau) = \frac{b^4}{16\pi^2\tau^2} \exp\left(-b^2 \frac{(x-y)^2}{4\tau}\right) \exp\left(-\left[\frac{(\frac{\lambda_2}{4}\varphi_2^2 + \lambda_3\varphi_1^2)^2}{\frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2} + (\lambda_3\varphi_1\varphi_2)\right] \frac{\tau}{b^2}\right) \quad (5.36)$$

and then plugging this into the Mellin transform for the zeta function we get a similar result for the effective potential:

$$\begin{aligned} V(\varphi_1, \varphi_2) &= -\frac{b^4}{32\pi^2} \frac{d}{ds} \left\{ \frac{1}{(s-2)(s-1)} \left(\frac{(\frac{\lambda_2}{4}\varphi_2^2 + \lambda_3\varphi_1^2)^2}{\frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2} + (\lambda_3\varphi_1\varphi_2) \right)^{2-s} \right\} \Big|_{s=0} \\ &= \frac{1}{64\pi^2} \left[\frac{(\frac{\lambda_2}{4}\varphi_2^2 + \lambda_3\varphi_1^2)^2}{\frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2} + (\lambda_3\varphi_1\varphi_2) \right]^2 \left(-\frac{3}{2} + \ln \frac{\left[\frac{(\frac{\lambda_2}{4}\varphi_2^2 + \lambda_3\varphi_1^2)^2}{\frac{\lambda_1}{4}\varphi_1^2 + \lambda_3\varphi_2^2} + (\lambda_3\varphi_1\varphi_2) \right]}{b^2} \right) \end{aligned} \quad (5.37)$$

The quantities φ_1, φ_2 should be interpreted as the classical fields. Let

$$\hat{A} = -\partial^2 + m_1^2 + \left(\frac{\lambda_1}{4} + \lambda_3 \tan \beta\right) \phi_1^2.$$

Going through the same procedure as before and knowing that

$$g = h_1 + \frac{\tan \beta}{\frac{\lambda_1}{4} + \lambda_3 \tan \beta} h_2$$

is a field redefinition and the effective potential definition will remain unchanged. The first field gives us the less complicated effective potential than the h_2 case where the off-diagonal terms contribute as a $\varphi_1\varphi_2$ mixing term.

The effective potential computed at one-loop gives us a degenerate ground state but is not at the origin as it is classically. This means physically that the symmetry of the

theory has been broken. These methods could be relevant to other more realistic two-Higgs models such as the Minimal Supersymmetric Standard Model.

CHAPTER 6

APPLICATIONS OF ZETA FUNCTION METHOD

6.1 Coleman-Weinberg in Type 0B String theory

Type 0B string theory is a tachyonic string theory in ten dimensions with no spacetime-like supersymmetry and is often considered to be a toy model. The open string sector contains D-brane degrees of freedom and according to the amount of D-branes present we get a world-volume gauge theory with gauge groups $U(N)$. There is an exchange of a string amongst the branes and there is an interaction energy. The interaction energy gets loop correction just as it does in other theories [57, 58, 59, 60]:

$$S = \frac{1}{g} \int d^4x \text{tr} \left\{ -\frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi^i)^2 - \frac{1}{2} [\Phi^i, \Phi^j]^2 \right\}. \quad (6.1)$$

The scalar potential has a degenerate set of minima:

$$\Phi_{cl}^i = \text{diag}(y_a^i), \quad a = 1, \dots, N. \quad (6.2)$$

The coordinates y_a^i , $i = 1, \dots, 6$ describe positions of N parallel static three-branes in nine-dimensional space. Since the potential does not depend on y_a^i , D-branes do not inter-

act at the classical level:

$$\Gamma = 4\text{Tr} \ln (-\partial^2 + Y^2) = \text{Vol} 4 \int \frac{d^4 p}{(2\pi)^4} \text{tr} \ln (p^2 + Y^2) \quad (6.3)$$

$$= \text{quadratically divergent term} + \text{Vol} \frac{1}{8\pi^2} \text{tr} Y^4 \ln \frac{Y^2}{M^2} . \quad (6.4)$$

The effective potential is

$$V(r) = \frac{1}{4\pi^2} r^4 \ln \frac{r^2}{\Lambda^2}, \quad (6.5)$$

where M is an UV cutoff. The quadratic and the logarithmic divergences in the effective action should be cancelled by appropriate counter terms.

6.2 Gross-Neveu Model

The Lagrangian for the Gross-Neveu model is

$$\mathcal{L}_{GN} = \bar{\psi}_i i \not{\partial} \psi_i + \frac{g^2}{2} (\bar{\psi}_i \psi_i)^2 . \quad (6.6)$$

The classical Lagrangian has a discrete chiral symmetry. Let $\Lambda^2 = \mu^2 \exp(\frac{-2\pi}{Ng^2})$

$$\psi_i \rightarrow \gamma_5 \psi_i \quad \bar{\psi}_i \rightarrow -\bar{\psi}_i \gamma_5 \quad (6.7)$$

Fermionic (Grassman-valued) fields are not considered as classical field theories, for example, as possible background fields for a quantum field theory calculation. However, fermionic quantum fields can pair up and form a composite bosonic field $\sigma \sim \bar{\psi} \psi$, which can attain a vacuum expectation value. The Gross-Neveu Lagrangian can now be written

as

$$\tilde{\mathcal{L}} = \bar{\psi}_i \not{\partial} \psi_i - \frac{1}{2g^2} \sigma^2 - \sigma \bar{\psi}_i \psi_i \quad (6.8)$$

and

$$\begin{aligned} Z &= \int \prod_i \mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i \mathcal{D}\sigma e^{\imath S(\sigma, \psi_i, \bar{\psi}_i)} = \int \prod_i \mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i \mathcal{D}\sigma e^{\imath \int d^2x \bar{\psi}_i (\not{\partial} + \sigma) \psi_i - \frac{\sigma^2}{2g^2}} \\ &= \int \mathcal{D}\sigma e^{\imath \int d^2x \frac{-\sigma^2}{2g^2}} \det(\not{\partial} + \sigma)^N \\ &= \int \mathcal{D}\sigma e^{\imath \int d^2x \mathcal{L}(\sigma)} \end{aligned} \quad (6.9)$$

with

$$\mathcal{L}(\sigma) = -\frac{\sigma^2}{2g^2} + \imath N \log \text{Det}(\not{\partial} + \sigma) . \quad (6.10)$$

The zeta function method is used here so we may as well guess the value of the determinant using the heat equation:

$$\begin{aligned} \log \det(\not{\partial} + \sigma) &= \int \frac{d^2p}{(2\pi)^2} \log \det(\not{p} + \sigma) \\ &= \int \frac{d^2p}{(2\pi)^2} \log \det \begin{pmatrix} \sigma & -\imath p_0 + \imath p_1 \\ \imath p_0 + \imath p_1 & \sigma \end{pmatrix} \\ &= \int \frac{d^2p}{(2\pi)^2} \log(\sigma^2 - p^2) . \end{aligned} \quad (6.11)$$

To compute the effective potential of the Gross-Neveu model we use the same method as that of the ϕ^4 theory except now we are living in two dimensions with a classical field σ_{cl} . We computed the zeta function for this interaction term and its determinant.

Using $\ln \det A = \zeta'(s)|_{s=0}$ we follow the pattern of the four dimensional theory to

evaluate the quantity for the effective potential,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int d\tau \tau^{s-1} \frac{1}{4\pi\tau} e^{-B\tau} , \quad (6.12)$$

where $B = b^2/\frac{1}{2}\sigma^2$ and the integral is computed and results in the effective potential,

$$\begin{aligned} V_{eff}(\sigma_{cl}) &= \frac{\sigma_{cl}^2}{2g^2} + \frac{N}{4\pi} \sigma_{cl}^2 \left(\log \frac{\sigma_{cl}^2}{\mu^2} - 1 \right) \\ &= \frac{N\sigma_{cl}^2}{4\pi} \left(\log \frac{\sigma_{cl}^2}{\Lambda^2} - 1 \right) . \end{aligned} \quad (6.13)$$

From these examples we note that the whole approach of the zeta function is ubiquitous and adds to the bag of tricks if one needs to calculate the one loop effects in a theory. It works the same way in other fields with different spins. I have only done this for the scalar cases in two dimensions, four dimensions and in type 0B string theory. It is of the Coleman-Weinberg type, meaning that it is pure quantum mechanical construct.

6.3 Standard Model Effective Potential

For completeness I include the effective potential without calculating their heat kernel as it is straight forward and results are standard [56]. The following effective potential is easily extended from consideration of spin of the particle which affects the sign,

$$V = V_0 + V_{vector} + V_{scalar} + V_{fermions} , \quad (6.14)$$

where

$$V_0 = \frac{1}{2}\mu^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 , \quad (6.15)$$

$$V_{vector} = \frac{3[2g^4 + (g^2 + g'^2)^2]}{1024\pi^2} \phi_c^4 \ln \frac{\phi_c^2}{M^2} , \quad (6.16)$$

$$V_{scalar} = \frac{1}{64\pi^2} (\mu^2 + 3\lambda\phi^2)^2 \ln\left(\frac{\mu^2 + 3\lambda\phi^2}{M^2}\right) + \frac{3}{64\pi^2} (\mu^2 + \lambda\phi^2) \ln\left(\frac{\mu^2 + \lambda\phi^2}{M^2}\right) , \quad (6.17)$$

$$V_{fermions} = -\frac{2\lambda^4}{64\pi^2} \phi_c^4 \ln\left(\frac{\phi_c^2}{M^2}\right) . \quad (6.18)$$

6.4 Super-Feynman Rules

I will only deal with $N = 1$ supersymmetry since this case applies to the Higgs sector of the Standard Model. The Higgs are chiral superfields. The papers that helped in this thesis are [18] [3], as well as the texts [10, 16, 4], and texts on supersymmetry [6, 9, 8, 7]. I shall use the notation of the book [9] and is summarized in the end of this work.

Supersymmetry is a symmetry between states of bosons and fermions. Schematically,

$$Q|B\rangle = |F\rangle , \quad Q|F\rangle = |B\rangle \quad (6.19)$$

to denote how a supersymmetric charge act on the graded superspace states and where Q is a supersymmetry generator of spin 1/2.

The $N = 1$ Lagrangian is determined from the following data.

1. Superpotentials W
2. Kähler K
3. Gauge Kinetic function f

These quantities have an effective treatment meaning that it is calculable. Supersymmetric gauge field theory is built on the idea of superbundles.

6.5 Supersymmetry

We will use the two component notation for it simplifies most of the work and not follow the older ones as projections appear all over the place.

Supersymmetry algebra is a \mathbb{Z}_2 -graded module with the properties,

$$\{Q_A, \bar{Q}_{\dot{A}}\} = -2(\sigma^\mu)_{A\dot{A}}P_\mu, \quad (6.20)$$

$$\{Q_A, Q_B\} = \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0, \quad (6.21)$$

$$[P_\mu, Q_\alpha] = 0. \quad (6.22)$$

6.5.1 Superspace and Superfields

Let $(x^\mu, \theta^A, \theta_{\dot{A}})$ which is denoted as $M^{3,1|1}$, M will be \mathbb{R} . Since the MSSM is $N = 1$ extension of the SM, we will concentrate on this, however other extended supersymmetries are still important.

Let P be a principal G -bundle over a manifold M with an allowed variation of dimensionality which is auxiliary and needed for Dimensional regularisation. G is a compact simple Lie groups which is $SU(3) \times SU(2) \times U(1)$. The spacetime gets an addition of grassmanian coordinates making it into a superspace, $M^{3,1|1}$ which locally is $(x^\mu, \theta_A, \bar{\theta}_{\dot{A}})$. Fields on this space are called superfields and its expansion terminates in θ variables.

The field content is four scalars f, M, N , and D along with vectors A and spinors ξ and η which are lefthanded and the righthanded $\bar{\chi}$ and $\bar{\lambda}$ making a total of sixteen real bosonic and sixteen real fermionic only off shell.

$$\mathcal{F}(x, \theta, \bar{\theta}) = f(x) + \sqrt{2}\theta\xi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) \quad (6.23)$$

$$+ \theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\eta(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \quad (6.24)$$

where $\theta\theta \equiv \theta^\alpha\theta_\alpha$.

$$(x^\mu, \epsilon, \bar{\epsilon}) \rightarrow (x^\mu - i\theta\sigma^\mu\bar{\epsilon} + i\epsilon\sigma^\mu\bar{\theta}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) \quad (6.25)$$

The infinitesimal supertransformations are, by applying

$$i(\epsilon Q + \bar{\epsilon}Q) \quad (6.26)$$

applied to equation (6.23),

$$\delta f = \sqrt{2}\epsilon\xi + \sqrt{2}\bar{\epsilon}\bar{\chi}, \quad (6.27)$$

$$\delta(\sqrt{2}\xi_A) = 2\epsilon_A M + (\sigma^\mu\bar{\epsilon})_A(-i\partial_\mu f + A_\mu), \quad (6.28)$$

$$\delta(\sqrt{2}\bar{\chi}_A) = 2\bar{\epsilon}^A N - (\bar{\sigma}^\mu\epsilon)^A(i\partial_\mu f + A_\mu), \quad (6.29)$$

$$\delta M = \bar{\epsilon}\bar{\lambda} + \frac{i}{\sqrt{2}}\partial_\mu\chi\sigma^\mu\bar{\epsilon}, \quad (6.30)$$

$$\delta N = \epsilon \eta - \frac{i}{\sqrt{2}} \epsilon \sigma^\mu \partial_\mu \bar{\chi}, \quad (6.31)$$

$$\delta A_\mu = \epsilon \sigma \bar{\lambda} + \eta \sigma_\mu \bar{\epsilon} + \frac{i}{\sqrt{2}} \partial_\mu \xi + \frac{i}{\sqrt{2}} \partial_\mu \bar{\chi} \bar{\epsilon} \quad (6.32)$$

$$+ i\sqrt{2} \epsilon \sigma_{\mu\nu} \partial^\nu \xi - i\sqrt{2} \bar{\epsilon} \sigma_{\mu\nu} \partial^\nu \bar{\chi}, \quad (6.33)$$

$$\delta \bar{\lambda}^{\dot{A}} = \bar{\epsilon}^{\dot{A}} D - \frac{i}{2} \bar{\epsilon}^{\dot{A}} \partial^\mu A_\mu - i(\bar{\sigma}^\mu \epsilon)^{\dot{A}} \partial_\mu M + (\bar{\sigma}^{\mu\nu} \bar{\epsilon})^{\dot{A}} \partial_\mu A_\nu, \quad (6.34)$$

$$\delta \eta_A = \epsilon_A D + \frac{1}{2} \epsilon \partial^\mu A_\mu - (\sigma^\mu \bar{\epsilon})_A \partial_\mu N - (\sigma^{\mu\nu} \epsilon)_A \partial_\mu A_\nu, \quad (6.35)$$

$$\delta D = i \partial_\mu (\eta \sigma^\mu \bar{\epsilon} + \bar{\lambda} \bar{\sigma}^\mu \epsilon). \quad (6.36)$$

Rather than go into details and proofs of formulae (6.27)–(6.36), I shall give the results and relate to sources cited [2].

The anticommuting variables, θ , satisfy the relations,

$$\theta \bar{\theta} + \bar{\theta} \theta = 0, \quad \theta^2 = 0, \quad \bar{\theta}^2 = 0 \quad (6.37)$$

$$Q_A = -i(\partial_A + i\sigma_{A\dot{B}}^\mu \bar{\theta}^{\dot{B}} \partial_\mu), \quad (6.38)$$

$$\bar{Q}^{\dot{A}} = -i(\bar{\partial}^{\dot{A}} + i\theta^B \sigma_{B\dot{B}}^\mu \epsilon^{\dot{B}\dot{A}} \partial_\mu), \quad (6.39)$$

or

$$\bar{Q}^{\dot{A}} = -i(\bar{\partial}^{\dot{A}} + i\bar{\sigma}^{\mu\dot{A}B}\theta_B\partial_\mu). \quad (6.40)$$

These are needed to make the superfield transformation manifest.

6.6 Chiral and Vector Multiplets and Superfields

Using superfields are an economical method to perform computation in supersymmetric extensions of known models.

6.6.1 Chiral Superfields

Let $\mathcal{F} \equiv \Phi$ to conform with other works, then the definition of a chiral superfield is

$$\mathcal{D}_{\dot{A}}\Phi = 0 \quad (6.41)$$

for the case of chiral superfield condition where the antichiral case is $\mathcal{D}_A\Phi = 0$. These mean that the respective field are not dependent of the dotted or undotted superspace coordinates. Also needed is the supercovariant derivatives,

$$\mathcal{D}_{\dot{A}} = \bar{\partial}_{\dot{A}} + i\theta^B\sigma_{B\dot{A}}^\mu\partial_\mu \quad (6.42)$$

and

$$\mathcal{D}_A \equiv \partial_A - i\sigma_{A\dot{B}}^\mu\theta^{\dot{B}}\partial_\mu. \quad (6.43)$$

6.7 Minimal Supersymmetric Standard Model

The particle content of the model is based on the Standard Model of $SU(3) \times SU(2) \times U(1)$ with minimal amount of superpartners and SUSY being broken at for example LHC scale. [2]

$$S = \int d^8z \left(\Phi_i^\dagger e^{2gt_i} \Phi_i + \eta V \right) + \int d^6z \left\{ \frac{1}{4} W^A W_A + \mathcal{W}(\Phi_i) \right\} + \int d^6\bar{z} \left\{ \frac{1}{4} \bar{W}^{\dot{A}} \bar{W}_{\dot{A}} + \mathcal{W}^\dagger(\Phi_i^\dagger) \right\} \quad (6.44)$$

is the Abelian gauge theory action.

We need to work in the case of non-abelian groups in order to get the Standard Model for low energy. The $SU(2)_L$ doublet in this thesis, the Minimal supersymmetric model(MSSM) will be used to study the electroweak symmetry and the appearance of the Higgs and other possible particles at specific scales, say, TeV.

In the MSSM, there are two Higgs scalar fields, H_u and H_d and are complex doublets. The scalar potential using standard techniques was derived in [32] [33]. However it can be shown that the same results can be achieved in using the zeta function method:

$$V_{eff}(A, F) = +\frac{\hbar}{64\pi^2} \text{Tr}(X^4) \ln X^2 \quad (6.45)$$

$$S[\Phi] = \int d^4x \left(-\frac{1}{2} \bar{D}D \right) \left\{ \Phi_a^\dagger \left(\frac{1}{2} \bar{D}D \right) \Phi_a \left(\lambda_a \Phi_a + \frac{1}{2} m_{ab} \Phi_a \Phi_b + \frac{1}{2} g_{abc} \Phi_a \Phi_b \Phi_c + \text{hc} \right) \right\} . \quad (6.46)$$

CHAPTER 7

CONCLUSION

Electroweak spontaneous symmetry breaking is essential for generating masses in the theory from the vacuum expectation value of the Higgs field. The resulting theory is renormalisable, unitary, and provides a description of low-energy phenomena in good agreement with experiment. It also permits extensions to include new fields and interactions, such as the Minimal Supersymmetric Standard Model (MSSM).

The effective potential allows loop (radiative) corrections to be included in studying the vacuum of the theory to determine if it induces symmetry breaking. At one-loop level, the effective potential is given by the determinant of an elliptic operator (the Laplacian on a four-sphere). This should be contrasted with the more common approach of a loop expansion in terms of Feynman diagrams. As in all loop calculations, a method is needed to regularise the divergences that occur in quantum field theory. The zeta function method used in this thesis is particularly elegant and simple. The mathematical formalism for calculating the determinant of elliptic operators is developed in detail to provide a mathematical justification of the applicability of the zeta function method.

One of the key results of this thesis is the application of the zeta function method to a model containing two interacting scalar fields without any assumptions of symmetries such as in $O(N)$ -symmetric multiple-scalar models. Application of the zeta function method to the two-scalar model requires diagonalization of the quadratic terms in the two scalar fields. Application of the zeta function method to other models in quantum field

theory and string theory have also been presented in this thesis, demonstrating that the zeta function method is as ubiquitous in physics as it is in mathematics.

REFERENCES

- [1] P. Cartier, A Mad Day's Work: From Grothendieck To Connes and Kontsevich, The Evolution of Concepts of Space and Symmetry, Bull. Am. Math. Soc. 38 (2001) 389.
- [2] M. Drees, R.M. Godbole, P. Roy, Theory and Phenomenology of Sparticles, World Scientific, Singapore, 2004.
- [3] S.P. Martin, A supersymmetry Primer, arXiv:hep-ph/9709356.
- [4] N. Polonsky, Supersymmetry Structure and Phenomena: Extensions of the Standard Model, Springer, New York, 2001.
- [5] W. Waldschmidt, P. Moussa, J. M. Luck, C. Itzykson, From Number Theory to Physics, Springer-Verlag, New York, 1992.
- [6] P. West, Introduction to Supersymmetry and Supergravity, World Scientific, Singapore, 1990.
- [7] T. Morii, C.S. Lim, S.N. Mukherjee, The Physics of the Standard Model and Beyond, World Scientific, Singapore, 2004.
- [8] S.P. Misra, Introduction to Supersymmetry and Supergravity, Wiley-Interscience, Singapore, 1992.
- [9] J. Wess, J. Bagger, Supersymmetry and Supergravity, Princeton University Press, Princeton, 1992.
- [10] S. Weinberg, The Quantum Theory of Fields Vol. 3: Supersymmetry, Cambridge University Press, Cambridge, 2005.
- [11] P. Deligne *et al.*, Quantum fields and Strings: A course for mathematicians Vols. 1–2, AMS, Providence, 1999.
- [12] B. Hatfield, Quantum Field Theory of Point Particles and Strings, Addison-Wesley, Redwood City, 1992.
- [13] V. P. Nair, Quantum Field Theory: A Modern Perspective, Graduate Texts in Contemporary Physics Series, Springer, New York, 2005.
- [14] R. Ticciati, Quantum Field Theory for Mathematicians, Cambridge University Press, Cambridge, 1999.
- [15] A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press, Princeton, 2003.
- [16] R.N. Mohapatra, Unification And Supersymmetry: The Frontiers Of Quark - Lepton Physics, Springer, New York, 2003.

- [17] J. Glimm, A.M. Jaffe, Quantum Physics. A Functional Integral Point Of View, Springer, New York, 1981.
- [18] D.I. Kazakov, Supersymmetry in particle physics: The renormalization group viewpoint, Phys. Rep. 344 (2001) 359 [arXiv:hep-ph/0001257].
- [19] A. Sen, Radiative Corrections In Supersymmetric Gauge Theories, Phys. Rev. D 31 (1985) 2100.
- [20] J. Polchinski, String theory Vol. 2: Superstring theory and beyond, Cambridge University Press, Cambridge, 1998.
- [21] O. DeWolfe, S.B. Giddings, Scales and hierarchies in warped compactifications and brane worlds, arXiv:hep-th/0208123.
- [22] S. Donaldson, P.B. Kronheimer, Geometry of Four Manifolds, Oxford University Press, Oxford, 1990.
- [23] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer-Verlag, New York, 1992.
- [24] H.B. Lawson, M-L. Michelson, Spin Geometry, Princeton University Press, Princeton, 1989.
- [25] M. Berger, P. Gauduchon, E. Mazet, Le Spectre d'une Variete Riemannienne, Lecture Notes in Mathematics 194, Springer-Verlag, New York, 1971.
- [26] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, Orlando, 1980.
- [27] S.R. Coleman, E. Weinberg, Radiative Corrections As The Origin Of Spontaneous Symmetry Breaking, Phys. Rev. D 7 (1973) 1888.
- [28] R. Jackiw, Functional evaluation of the effective potential, Phys. Rev. D 9 (1974) 1686.
- [29] D. Vassilevich, Heat kernel expansion: user's manual, Phys. Rep. 388 (2003) 279.
- [30] M. Kontsevich, D. Zagier, Periods in Mathematics Unlimited, Year 2001 and Beyond, Eds. B. Engquist and W. Schmidt, Springer, New York, 2001.
- [31] G. 't Hooft, M. Veltman, Regularization and Renormalization of gauge fields, Nucl. Phys. B 44 (1972) 189, 213.
- [32] M. Huq, Evaluation Of Effective Potential In Superspace, Phys. Rev. D 16 (1977) 1733.
- [33] M.T. Grisaru, F. Riva, D. Zanon, The One Loop Effective Potential In Superspace, Nucl. Phys. B 214 (1983) 465.

- [34] J.F. Gunion, H.E. Haber, G.L. Kane, S. Dawson, *The Higgs Hunter's Guide*, Westview, New York, 2000.
- [35] L. Randall, Two Higgs Models for Large Tan Beta and Heavy Second Higgs, JHEP 0802 (2008) 84 [arXiv:0711.4360 [hep-ph]].
- [36] J.F. Gunion, H.E. Haber, The CP-conserving two-Higgs-doublet model: The approach to the decoupling limit, Phys. Rev. D 67 (2003) 075019 [arXiv:hep-ph/0207010].
- [37] M. Carena, H.E. Haber, Higgs boson theory and phenomenology V, Prog. Part. Nucl. Phys. 50 (2003) 63 [arXiv:hep-ph/0208209].
- [38] S.G. Stephen, S.D. Miller, Riemann's zeta function and beyond, Bull. Amer. Math. Soc. 41 (2004) 59.
- [39] B. Poiline, A. Waldron, Automorphic Forms: a physicist's survey, in *Frontiers in Number Theory, Physics, and Geometry II*, Springer, Berlin, 2007.
- [40] V. Elias, R.B. Mann, D.G.C. McKeon, T.G. Steele, Radiative Electroweak symmetry breaking beyond leading logarithms, Can. J. Phys. 84 (2006) 545. [arXiv:hep-ph/0508107].
- [41] R. Arnowitt, S. Deser and C.W. Misner, The dynamics of general relativity, arXiv:gr-qc/0405109.
- [42] M. Daniel and C.M. Viallet, The Geometrical Setting Of Gauge Theories Of The Yang-Mills Type, Rev. Mod. Phys. 52 (1980) 175.
- [43] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.
- [44] J. Fuchs, C. Schweigert, *Symmetries, Lie Algebras and Representations*, Cambridge University Press, Cambridge, 1997.
- [45] T. Brocker, T.T. Dieck, *Representation of Compact Lie Groups*, GTM 98, Springer, New York, 1985.
- [46] G.E. Bredon, *Topology and Geometry* GTM 139, Springer, New York, 1993.
- [47] M. Spivak, *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*, Westview Press, New York, 1971.
- [48] W. Thirring, M.H. Evans, *Classical Mathematical Physics: Dynamical Systems and Field Theories*, 3rd edition, Springer, New York, 1997.
- [49] T.G. Steele, *Techniques of Theoretical Physics*, Physics 402 Lecture notes (unpublished).
- [50] G. Arfken, H. Weber, *Mathematical Methods for Physicists*, 6th Edition, Elsevier, New York, 2005.

- [51] J. Milnor, J. Stasheff, Characteristic Classes, Princeton University Press, Princeton, 1974.
- [52] A. Derdzinski, Geometry of the Standard Model of Elementary Particles, Springer-Verlag, New York, 1992.
- [53] G. McCabe, The Structure and Interpretation of the Standard Model, Elsevier, Amsterdam, 2007.
- [54] D.D. Bleecker, Physics From The G Bundle Viewpoint, Int. J. Theor. Phys. 23 (1984) 735.
- [55] K.D. Kennaway, Effective potentials and the vacuum structure of quantum field theories, arXiv:hep-th/0407084.
- [56] M. Sher, Electroweak Higgs Potentials And Vacuum Stability, Phys. Rep. 179 (1989) 273.
- [57] K. Zarembo, Coleman-Weinberg mechanism and interaction of D3-branes in type 0 string theory, Phys. Lett. B 462 (1999) 70 [arXiv:hep-th/9901106].
- [58] A.A. Tseytlin, K. Zarembo, Effective potential in non-supersymmetric $SU(N) \times SU(N)$ gauge theory and interactions of type 0 D3-branes, Phys. Lett. B 457 (1999) 77 [arXiv:hep-th/9902095].
- [59] R. Blumenhagen, A. Font, D. Lust, Non-supersymmetric gauge theories from D-branes in type 0 string theory, Nucl. Phys. B 560 (1999) 66 [arXiv:hep-th/9906101].
- [60] I.R. Klebanov, A.A. Tseytlin, D-branes and dual gauge theories in type 0 strings, Nucl. Phys. B 546 (1999) 155 [arXiv:hep-th/9811035].
- [61] E. Gildener, S. Weinberg, Symmetry Breaking And Scalar Bosons, Phys. Rev. D 13 (1976) 3333.
- [62] D.G.C. McKeon, T.N. Sherry, Operator Regularization and One Loop Green's Functions, Phys. Rev. D 35 (1987) 3854.
- [63] S. Pokorski, Gauge Field Theories, 2nd Ed., Cambridge University Press, Cambridge, 2000.
- [64] T.P. Cheng, L.F. Li, Gauge Theory Of Elementary Particle Physics, Oxford University Press, Oxford, 1984.
- [65] C. Itzykson, J.B. Zuber, Quantum Field Theory, Mcgraw-Hill, New York, 1980.
- [66] L.H. Ryder, Quantum Field Theory, Cambridge University Press, Cambridge, 1985.

- [67] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, 2nd Edition, Springer, New York, 1990.
- [68] D.B. Ray, I.M. Singer, Analytic Torsion for complex manifolds, *Ann. Math.* 98 (1973) 154.
- [69] E. Corrigan, P. Goddard, H. Osborn, S. Templeton, Zeta Function Regularization And Multi - Instanton Determinants, *Nucl. Phys. B* 159 (1979) 469.
- [70] R.J. Rivers, Path Integral Methods in Quantum Field Theory, Cambridge University Press, Cambridge, 1987.
- [71] L. Bailin, A. Love, Introduction to Gauge Field Theory, Taylor Francis, New York, 1993.
- [72] R.A. Bertlmann, Anomalies in Quantum Field Theory, Oxford, Oxford, 2000.
- [73] R. Forman, Functional Determinants and Geometry, *Invent. Math.* 88 (1987) 447.
- [74] M. Kontsevich, S. Vishik, Geometry of determinants of elliptic operators, [arXiv:hep-th/9406140](https://arxiv.org/abs/hep-th/9406140).
- [75] D.S. Freed, Determinants, Torsion, and Strings, *Commun. Math. Phys.* 107 (1986) 483.
- [76] J.M. Bismut, D.S. Freed, The Analysis of Elliptic Families 2: Dirac Operators, Eta Invariants, and The Holonomy Theorem, *Commun. Math. Phys.* 107 (1986) 103 .
- [77] M. Dine, Supersymmetry and String theory: Beyond the Standard Model, Cambridge, Cambridge, 2007.